Discrete Methods in Informatics	October 17, 2005
Lecture 2 (edge-coloring 2)	
Lecturer: Satoru Iwata	Scribe: Yasunori Horikoshi

In this course, we will cover some topics in combinatorial optimization. The textbook is [2].

1 Edge coloring of simple graph

Edge coloring of undirected graph $G = (V, E)$ is assigning color to each edge $e \in E$ so that any two edges having end-vertex in common have different colors. The minimum number of colors required for an edge coloring of G is denoted by $\gamma(G)$.

Edge coloring is a classical problem in graph theory, especially because proving the 4-color theorem is equivalent to showing the 3-edge-colorability of planar bridgeless cubic graphs.

For any simple graph G, the following inequality is trivial:

$$\Delta(G) \leq \gamma(G),$$

where $\Delta(G)$ denotes the maximum degree among the vertices.

If G is a bipartite graph, the inequality holds with equality (Lecture 1, Thorem 1). In contrast, there is a graph G such that $\Delta(G) < \gamma(G)$. Nevertheless, the next theorem has been established.

Theorem 1 (Vizing [3, 4])**.** *For any simple graph G, we have $\Delta(G) \leq \gamma(G) \leq \Delta(G) + 1$.*

We use the following lemma in the proof taken from [2, §28.1].

Lemma 2. *Let G be a simple graph and let v be a vertex such that v and all its neighbours have degree at most k, while at most one neighbour has degree precisely k. If $G - v$ is k-edge-colorable, then so is G.*

Proof. We prove this by induction on k. The case $k = 0$ is trivial. We may assume that precisely one neighbour of v has degree k and the others have degree $k - 1$. Because we can add new vertices and edges so that the condition holds.

Assume that $G - v$ is k-edge-colorable. We define X_i $(i = 1, 2, \ldots, k)$ to be the set of neighbours of v to which color i is not assigned. We now consider a k-edge-coloring of $G - v$ that minimizes $\sum_{i=1}^{k} |X_i|^2$. Then there is a number i $(1 \leq i \leq k)$ such that $|X_i| = 1$. Assume otherwise, and then we have the following assessment:

$$\sum_{i=1}^{k} |X_i| = 2\deg(v) - 1 < 2k.$$

This inequality implies that there are numbers i and j such that $|X_i| = 0$ and $|X_j| \geq 3$. We now define H to be the subgraph of $G - v$ which consists of all edges colored by i or j. For any vertex $s \in X_j$, the connected component of H containing s must be a path. We can interchange the colors of the edges along the path, which makes $|X_i|^2 + |X_j|^2$ smaller. This contradicts to the choice of the k-edge-coloring.

Thus, we can assume $X_k = \{u\}$ without loss of generality. Let G' be the subgraph of G obtained by deleting the edge uv and the edges of color k. Then $G' - v$ is $(k - 1)$-edge-colorable. In G', degrees of

v and its neighbours are less than or equal to $k-1$ and at most one neighbour has degree $k-1$. So G' is $(k-1)$-edge-colorable by the inductive assumption. Then, putting back the edges we deleted and assigning color k to the edge uv, we have k-edge-coloring of G. □

Now we prove the theorem.

Proof of Theorem 1. Let $k = \Delta(G) + 1$. Then, any vertex of G satisfies the condition of Lemma 2. Thus we can eliminate edges from G one by one until only one edge is left. Then the resulting graph is of course k-edge-colorable. Inductive applications of Lemma 2 accomplish the proof. □

The proof gives an algorithm for finding a $(\Delta(G) + 1)$-edge-coloring in $\mathrm{O}(\Delta n^2)$ time, where n is the number of vertices. Merging the vertices of degree less than $\Delta(G)/2$, we have $\Delta n = \mathrm{O}(m)$, where m is the number of edges. This implies that the algorithm runs in $\mathrm{O}(nm)$ time.

2 Coloring of complete graph

We have seen that for any simple graph G either $\gamma(G) = \Delta(G)$ or $\gamma(G) = \Delta(G) + 1$ holds. It is known that determining whether $\gamma(G) = \Delta(G)$ holds or not is NP-complete [1]. For some specific classes of graphs, however, we are able to determine $\gamma(G)$. For example, we have the following theorem on the complete graphs.

Theorem 3. *Let K_n be a complete graph of n vertices, then*

$$\gamma(K_n) = \begin{cases} n-1 & (n : \text{even}) \\ n & (n : \text{odd}) \end{cases}$$

holds.

Proof. First, we show that $\gamma(K_n) \leq n$ holds for any n. Let $V = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ be vertex set of K_n. Then assigning color $i + j \pmod n$ to edge $v_i v_j$ gives an edge coloring.

Next, we show that $\gamma(K_n) > n-1$ holds for any odd n. If K_n is $(n-1)$-edge-colorable then there are matchings $\{M_i\}_{i=0}^{n-2}$ such that $M_i \cap M_j = \emptyset \ (i \neq j)$ and $E = M_0 \cup M_1 \cup \ldots \cup M_{n-2}$. Since $|M_i| \leq (n-1)/2$, we have $|E| = \sum |M_i| \leq (n-1)^2/2$, which contradicts to $|E| = n(n-1)/2 > (n-1)^2/2$.

Finally, we show that $\gamma(K_n) = n-1$ for any even n. The subgraph $K_n - v_n$ is $(n-1)$-edge-colorable by the above method. Then the color which is not assigned to v_i is $2i \pmod{n-1}$. For each $i \in \{0, 1, 2, \ldots, n-2\}$ let $w_i := 2i \pmod{n-1}$. Since $n-1$ is odd, we have $i \neq j \Rightarrow w_i \neq w_j$. Thus, assigning w_i to edge $v_i v_{n-1}$ $(i = 0, 1, 2, \ldots, n-2)$, we obtain an $(n-1)$-edge-coloring of K_n. □

References

[1] I. Holyer: The NP-completeness of edge-coloring, *SIAM Journal on Computing* 10 (1981), 718–720.

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[3] V. G. Vizing: Ob otsenke khromaticheskogo klassa p-grafa [Russian; On an estimate of the chromatic class of a p-graph], *Diskretnyĭ Analiz* 3 (1964), 25–30.

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