Discrete Methods in Informatics November 28, 2005

Lecture 6 (Simultaneous Exchangeability)

Lecturer: *Satoru Iwata* *Scribe*: *Noriyuki Ohkawa*

Simultaneous Exchangeability

Proposition 1. *For any circuit C and cocircuit C^*, $|C \cap C^*| \neq 1$.*

Proof. We assume $|C \cap C^*| = 1$. Let e be the only vertex in $C \cap C^*$. Then $C - \{e\}$ is an independent set in $\mathcal{M}$, and $C^* - \{e\}$ is an independent set in $\mathcal{M}^*$.

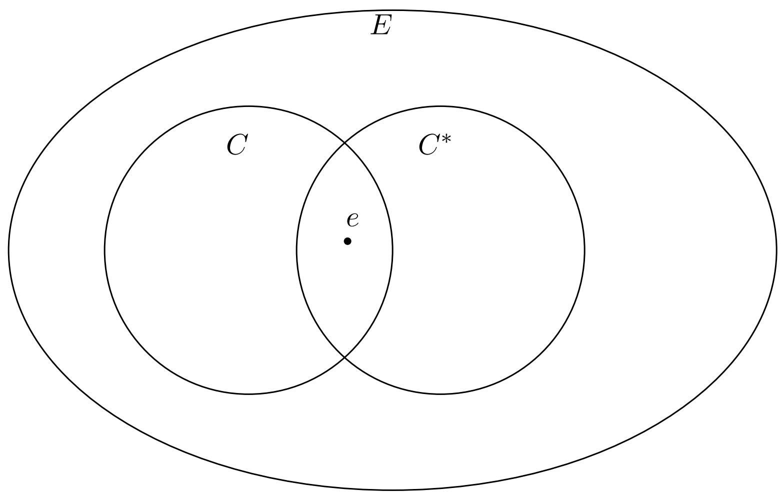

Figure 1: Like this

This mean that $E - C^* \backslash \{e\}$ includes a base of $\mathcal{M}$. Hence there exists $B \in \mathcal{B}$ such that $C \backslash \{e\} \subseteq B \subseteq E - C^* \backslash \{e\}$. Since $e \notin B$, We get $C^* \subseteq E \backslash B$. It contradicts to C^* is dependent set in $\mathcal{M}^*$ that C^* is included in $E \backslash B$ which is a base of $\mathcal{M}^*$.

□

Proposition 2. *For any $B_1, B_2 \in \mathcal{B}$ and any $b \in B_1 \backslash B_2$, there exists $e \in B_2 \backslash B_1$ such that $(B_1 \backslash \{b\}) \cup \{e\} \in \mathcal{B}, (B_2 \cup \{b\}) \backslash \{e\} \in \mathcal{B}$.*

Proof. For fundamental cocircuit $C^*(B_1|b) = \{e | (B_1 \backslash \{b\}) \cup \{e\} \in \mathcal{B}\}$ and fundamental circuit $C(B_2|b) = \{e | (B_2 \cup \{b\}) \backslash \{e\} \in \mathcal{B}\}$, $b \in C^*(B_1|b) \cap C(B_2|b)$. By Proposition1, we can see that $|C^*(B_1|b) \cap C(B_2|b)| > 1$. Thus, there exists $e \in (C^*(B_1|b) \cap C(B_2|b)) \backslash \{b\} \subseteq B_2 \backslash B_1$.

□

Minor

Definition 3. For matroid $\mathcal{M} = (E, \mathcal{I})$, and $F \subseteq E$, we define reduction $M \cdot F$ as matroid $(F, \mathcal{I}^F)$ which has $\mathcal{I}^F = \{I | I \in \mathcal{I}, I \subseteq F\}$ as family of independent set.

Definition 4. For $Z \subseteq E$, we define contraction $\mathcal{M}/Z$ as matroid $(E \backslash Z, \rho_Z)$ which has $\rho_Z(X) = \rho(X \cup Z) - \rho(Z) \;\; (X \subseteq E \backslash Z)$ as rank function.

There is relation $\mathcal{M}/Z = (\mathcal{M}^* \cdot (E \backslash Z))^*$ between reduction and contraction.

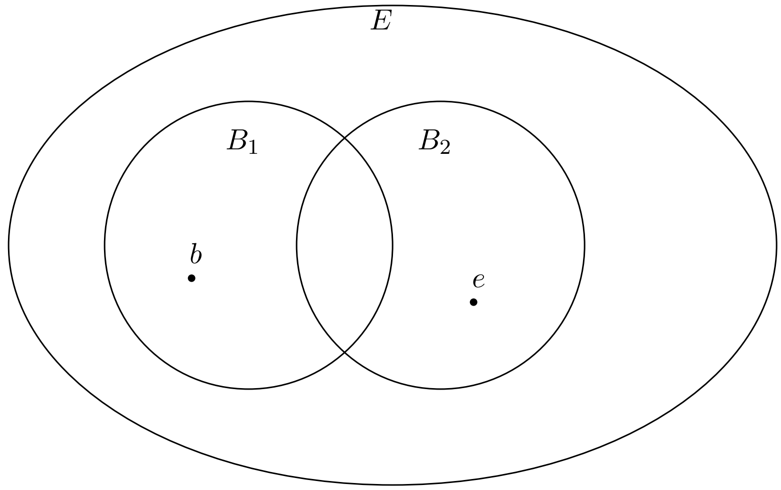

Figure 2: Simultaneous Exchangeability

Definition 5. We define minor as matroid generated by applying reduction and contraction repeatedly.

Properties which can express in terms of graph is closed under its minor.

Theorem 6 (Tutte). *$\mathcal{M}$ is binary matroid, iff $\mathcal{M}$ doesn't include $U_{4,2}$ as minor.*

Proof. Omitted. □

Since properties which can express in terms of graph is closed under its minor, if $\mathcal{L}$ is minor of binary matroid $\mathcal{M}$, $\mathcal{L}$ is binary matroid. Now $\mathcal{M}$ doesn't include uniform matroid as minor. Because $U_{4,2}$ is not binary matroid.

Exchangeable Graph

Definition 7. For matroid $\mathcal{M} = (E,\mathcal{I})$, $I \in \mathcal{I}$, $J \subseteq \mathrm{cl}(I)$ and $H = \{(i,j)|j \in J\backslash I, i \in C(I|j)\}$, we define exchangeable graph as graph $G(I,J) = (I\backslash J, J\backslash I; H)$.

Proposition 8. *$B, D \in \mathcal{B} \Rightarrow G(B,D)$:exchangeable graph which have perfect matching.*

Proof. We now consider $B, D \in \mathcal{B}$ that have no perfect matching and minimize $|B\backslash D| = |D\backslash B|$. By simultaneous exchangeability, for any $j \in D\backslash B$, there exists $i \in B\backslash D$, $(i,j) \in H, D' = (D - \{i\}) \cup \{i\} \in \mathcal{B}$. Obviously $|B\backslash D'| = |B\backslash D| - 1$. Hence $G(B,D')$ have perfect matching. Therefore $M \cup \{(i,j)\}$ is perfect matching of $G(B,D)$. □

Proposition 9. *For $B \in \mathcal{B}$, if exchangeable graph $G(B,D)$ ($|B| = |D|$) have unique perfect matching, $D \in \mathcal{B}$.*

Proof. Assume that $D \notin \mathcal{B}$. There exists $C \in \mathcal{C}$ such that $C \subseteq D$. For any $i \in B\backslash D$, $|C \cap C^*(B|i)| \neq 1 \Rightarrow |(C\backslash B) \cap C^*(B|i)| \neq 1$. If i is connected to some node in $C\backslash B$ by perfect matching M, $|(C\backslash B) \cap C^*(B|i)| > 1$. Hence, every these nodes have 2 or more edges witch connect to $C\backslash B$. Since there exists alternating closed path in $G(B,D)$, there exists another matching. But it contradict uniqueness of perfect matching. □

Claim 10. *For $I \in \mathcal{I}, |I| = |J|, J \subseteq \mathrm{cl}(I)$, if $G(I,J)$ have unique perfect matching, $j \in \mathcal{I}, \mathrm{cl}(I) = \mathrm{cl}(J)$.*

Proof. We get $J \in \mathcal{I}$ by applying above property to $\mathcal{M} \cdot \mathrm{cl}(I)$.
Since $\mathrm{cl}(J) \subseteq \mathrm{cl}(\mathrm{cl}(I)) = \mathrm{cl}(I)$, and J is base of $\mathcal{M} \cdot \mathrm{cl}(I)$, $\mathrm{cl}(J) = \mathrm{cl}(I)$. □