1 Nash-Williams Theorem

Let \( G = (V, E) \) be an undirected graph and let \( d(v) \) be a degree of \( v \in V \). Given a mapping function \( b : v \to \mathbb{Z} \) which satisfies \( b(v) \leq d(v) \) for \( v \in V \), we define a \( b \)-detachment of \( G \) as follows:

**Definition 1.1. \( b \)-detachment**

Divide \( v \in V \) into \( b(v) \) nodes, and connect each edge to the nodes which are originally connected in \( G \).

Figure 1 shows the example of \( b \)-detachment.

Nash-Williams showed the following theorem about \( b \)-detachment.

**Theorem 1.2. Nash-Williams (1985)**

There exists a connected \( b \)-detachment if and only if \( \forall X \subseteq V, b(X) + c(G \setminus X) \leq e(X) + 1 \)

where \( b(X) = \sum_{v \in X} b(v) \), \( c(G \setminus X) \) denotes the number of components of \( G \setminus X \), and \( e(X) \) denotes the number of edge connecting to the node in \( X \).

Figure 2 shows the example of edges connecting to \( X \).

First we show the necessary condition briefly.

**Proof.** (Necessity)

Let \( G = (V, E) \) be a connected undirected graph after \( b \)-detachment. Let \( G' = (V', E') \) be an undirected
Figure 2: Example of edges connecting to $X$

Figure 3: Example of the proof of Necessity

Graph after contraction of each connected component of $G \setminus X$ (Figure 1). Then

$$|V'| = b(X) + c(G \setminus X) \quad (1)$$

and

$$|E'| = e(X) \quad (2)$$

hold. Since $G'$ is connected, we need $b(X) + c(G \setminus X) \leq e(X) + 1$. (If equality holds, $G'$ is a spanning tree).

We then give the complete proof of the Nash-Williams theorem.

**Proof.** Let $\Gamma = (W, \Gamma(E))$ be the graph obtained from $G$ by replacing each vertex $v \in V$ by $b(v)$ copies, and by connecting for each edge $e = uv$, the $b(u)$ new vertices associated with $u$ with $b(v)$ new vertices associated with $v$ (Figure 4).
We assign to these edges of $\Gamma$ the color $e'$. Then the connected $b$-detachment is obtained by the connected spanning tree of $\Gamma$ in which all the edges have different colors.

In summary, we have the following equivalent propositions.

- There exists a connected $b$-detachment exist.
- There exists a spanning tree of $\Gamma$ in which all the edges have a different color.
- There exist the bases of a graph matroid of $\Gamma$ which are also the independent sets of a partition matroid of color assignment. (The rank of base of a graph matroid is $|W| - 1$. Since we use at most one edge of each color in the spanning tree, these edges correspond to the independent sets of the partition matroid.)

**Proposition 1.3.** There exist the bases of a graph matroid of $\Gamma$ which are also the independent sets of a partition matroid of color assignment, if and only if $\forall F \subseteq E, \Gamma(E) \setminus \Gamma(F)$ have at most $|F| + 1$ components.

**Proof.** Let $\pi$ be the rank function and $\mathcal{I}$ be the independent set of the partition matroid. Let $\rho$ be the rank function and $\mathcal{F}$ be the independent set of the graph matroid of $\Gamma$. From the theorem in 7th lecture, we get

$$\max\{|I|\mid I \in \mathcal{I} \cap \mathcal{F}\} = \min\{\pi(X) + \rho(E - X)\mid X \subseteq E\} \tag{3}$$

Let us consider the bipartite graph in which the vertices are $E$ and $\Gamma(E)$. Figure 5 shows the example of the bipartite graph. In figure 5, whatever we choose $Y$ as a gray region and a solid line region, we have $\pi(Y) = |F|$. This equals the rank of partition matroid of $\Gamma(F)$, that is, $\pi(\Gamma(F)) = |F|$. We can therefore assume that $Y$ is equal to the $\Gamma(F)$. Then we have

$$\max\{|I|\mid I \in \mathcal{I} \cap \mathcal{F}\} = \min\{|F| + \rho(\Gamma(E) \setminus \Gamma(F))\mid F \subseteq E\} \tag{4}$$

Let $I \in \mathcal{I} \cap \mathcal{F}$ be the base of the graph matroid of $\Gamma$ and the independent set of the partition matroid, which satisfies $|I| = |W| - 1$. Therefore

$$\forall F \subseteq E, \rho(\Gamma(E) \setminus \Gamma(F)) + |F| \geq |W| - 1. \tag{5}$$

Let $c(X)$ represent the component of $X$. We have

$$\rho(\Gamma(E) \setminus \Gamma(F)) = |W| - c(\Gamma(E) \setminus \Gamma(F)). \quad \tag{6}$$
which follows
\[ \forall F \subseteq E, c(\Gamma(E) \setminus \Gamma(F)) \leq |F| + 1. \quad (7) \]

In summary, there exist a connected $b$-detachment if and only if $\forall F \subseteq E, \Gamma(E) \setminus \Gamma(F)$ has at most $|F| + 1$ components.

To proof the Nash-Williams theorem, we need to show that $\forall F \subseteq E, c(\Gamma(E) \setminus \Gamma(F)) \leq |F| + 1$ if and only if $\forall X \subseteq V, b(X) + c(G \setminus X) \leq e(X) + 1$.

Let $H_F$ and $I_F$ denote
\[ H_F : \Gamma(E) \setminus \Gamma(F) \]
\[ I_F : \text{The set of isolated vertices of } G \setminus F. \]

The isolated vertices of $c(G \setminus F)$ appears twice in $c(G \setminus F)$ and the number of a component parts which are not isolated vertcies is not changed. Therefore we have
\[ c(H_F) = c(G \setminus F) + b(I_F) - |I_F|. \quad (8) \]
We also have
\[ c(\Gamma(E) \setminus \Gamma(F)) \leq |F| + 1, \forall F \subseteq E. \] (9)

Using the equations, we have
\[ c(G \setminus F) - |F| + b(I_F) - |I_F| \leq 1, \forall F \subseteq E. \] (10)

Let \( F' \) be a set of edges which make \( I_F \) the isolated points, that is
\[ I_{F'} = I_F \setminus F. \] (11)

Using \( F' \), (10) is equivalent to
\[ c(G \setminus F') - |F'| + b(I_F) - |I_F| \leq 1, \forall F \subseteq E. \] (12)

Let \( e(X) \) denote the number of edge connecting to \( X \). Since \( c(G \setminus I_F) - |I_F| = c(G \setminus I_{F'}) = c(G \setminus I_F) \) and \( |F'| := e(I_{F'}) \), we have
\[ c(G \setminus I_F) - e(I_F) + b(I_F) \leq 1, \forall F \subseteq E. \] (13)

Since we can assume \( I_F \) for \( \forall F \), we define \( I_F \) as \( X \),
\[ c(G \setminus X) - e(X) + b(X) \leq 1, \forall X \subseteq V \] (14)

\[ \square \]

2 Generalized Nash-Williams Theorem

We have the following theorem as a generalized version of Nash-Williams theorem.

**Theorem 2.1.** If there exists a connected \( b \)-detachment of \( G \) then there exist a connected \( b \)-detachment of \( G \), in which the degree sequence is equivalent to any degree sequences \( \sigma(v) \), where a degree sequence \( \sigma(v) \) satisfies \( \sigma(v) = (\sigma_1(v) \ldots \sigma_k(v)) \), \( k = b(v) \), \( \sum_{i=1}^{k} \sigma_i(v) = d(v) \).

**Proof.** For \( \Gamma \), we assume that the connected spanning graph \( Q \) which uses each color just once and \( \sum_{s \in W} |d_Q(s) - \sigma(s)| \) is minimum.

Suppose that the following equation does not satisfied.
\[ d_Q(s) = \sigma(s), \forall s \in W. \] (15)

Then, for some \( v \in V \), there exist copies of \( v, s, t \in W \) which satisfies
\[ d_Q(s) > \sigma(s) \] (16)
\[ d_Q(s) < \sigma(t). \] (17)

Let
\[ e \in Q : \text{forbidden edge } \iff s \text{ and } t \text{ are not connected in } Q \setminus e. \]
Let $w$ be a copy of $u$. Then $s$ connects at most one forbidden edge. Choose a edge $e$ connecting $s$ which is not forbidden edge. Remove $e$ from $Q$ and add $e'$ to $Q$ which is not in $Q$. Then $Q \setminus \{e\} \cap \{e'\}$ is connected. Now, since $d_Q(s)$ is decrease by one and $d_Q(t)$ is increased by one, $\sum_{s \in W} |d_Q(s) - \sigma(s)|$ is decreased by two. This contradicts the minimality of $\sum_{s \in W} |d_Q(s) - \sigma(s)|$.

Therefore $\sum_{s \in W} |d_Q(s) - \sigma(s)|$ is equal to 0. \qed
3 Extension of Euler’s Theorem

Let $G$ be a connected euler graph. Since $G$ is an euler graph, we can consider $b$-detachment which satisfies the following equation:

$$b(v) = \frac{d(v)}{2} \ (v \in V) \quad (18)$$

and each node has a degree as follows:

$$\sigma(v) = (2, 2, \ldots, 2) \quad (19)$$

Applying Nash-Williams’s theorem to $G$, we have

$$b(X) + c(G\setminus X) \leq e(X) + 1 \quad (20)$$

We also have

$$b(X) = \sum_{v \in X} \frac{d(v)}{2} = e(X) - \frac{d(X, E\setminus X)}{2} \quad (21)$$

Using the above equations we have

$$c(G\setminus X) \leq \frac{1}{2} d(X, E\setminus X) + 1. \quad (22)$$

Since we have equality only when $b$-detachment is a spanning tree. Since the $b$-detachmnet inlucde a cycle, we have

$$c(G\setminus X) < \frac{1}{2} d(X, E\setminus X) + 1, \quad (23)$$

which follows

$$2c(G\setminus X) \leq d(X, E\setminus X) \quad (24)$$

since $G$ is an Euler graph and $d(X, E\setminus X)$ is even.