Discrete Methods in Informatics	December 12, 2006

Lecture 8 (Nash-Williams Theorem)

Lecturer: Satoru Iwata *Scribe: Daiki Kojima, Daisuke Okanohara*

1 Nash-Williams Theorem

Let $G = (V, E)$ be an undirected graph and let $d(v)$ be a degree of $v \in V$. Given a mapping function $b : v \to \mathbf{Z}$ which satisfies $b(v) \le d(v)$ for $v \in V$, we define a b-detachment of G as follows:

Definition 1.1. *b-detachment*
Divide $v \in V$ into $b(v)$ nodes, and connect each edge to the nodes which are originally connected in G.

Figure 1 shows the example of b-detachment.

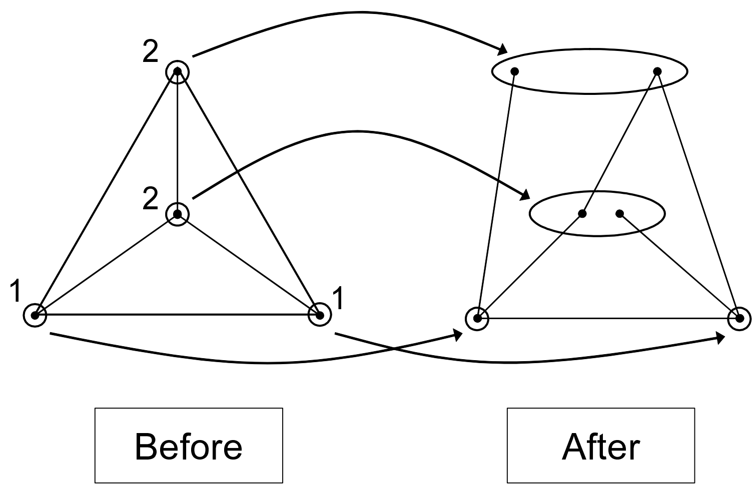

Figure.1 Example of b-detachment. The number aside of each node v denotes its $b(v)$.

Nash-Williams showed the following theorem about b-detachment.

Theorem 1.2. Nash-Williams (1985)

There exists a connected b-detachment if and only if $\forall X \subseteq V, b(X) + c(G \backslash X) \le e(X) + 1$

where $b(X) = \sum_{v \in X} b(v)$, $c(G \backslash X)$ denotes the number of components of $G \backslash X$, and $e(X)$ denotes the number of edge connecting to the node in X.

Figure 2 shows the example of edges connecting to X.

First we show the necessary condition briefly.

Proof. (Necessity)
Let $G = (V, E)$ be a connected undirected graph after b-detachment. Let $G' = (V', E')$ be an undirected

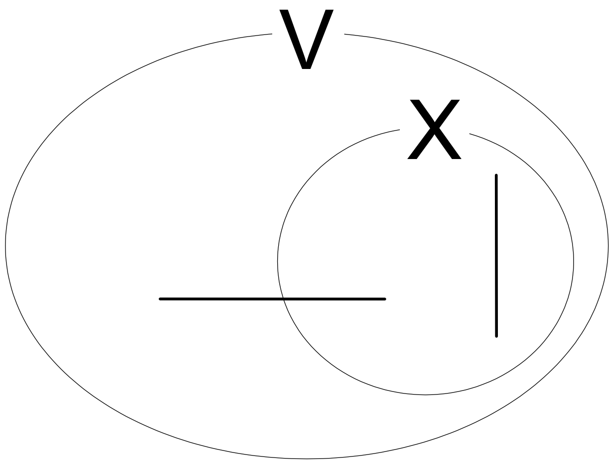

Figure.2 Example of edges connecting to X

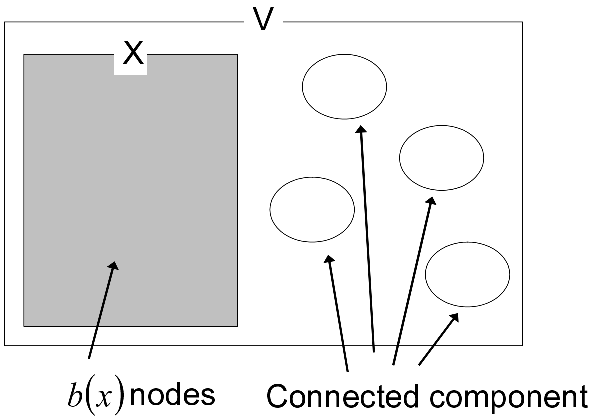

Figure.3 Example of the proof of Necessity

graph after contraction of each connected component of $G\backslash X$ (Figure 1). Then

$$|V'| = b(X) + c(G\backslash X) \tag{1}$$

and

$$|E'| = e(X) \tag{2}$$

hold. Since G' is connected, we need $b(X) + c(G\backslash X) \leq e(X) + 1$. (If equality holds, G' is a spanning tree).

□

We then give the complete proof of the Nash-Williams theorem.

Proof. Let $\Gamma = (W, \Gamma(E))$ be the graph obtained from G by replacing each vertex $v \in V$ by $b(v)$ copies, and by connecting for each edge $e = uv$, the $b(u)$ new vertices associated with u with $b(v)$ new vertices associated with v (Figure 4).

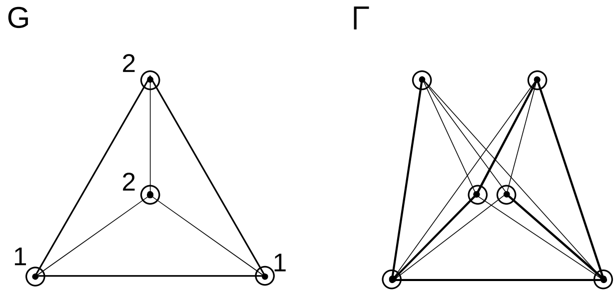

Figure.4 Original graph G (left) and its Γ obtained by replacing each vertex $v \in V$ by $b(v)$ copies and by connecting for each edge properly.

We assign to these edges of Γ the color e'. Then the connected b-detachment is obtained by the connected spanning tree of Γ in which all the edges have different colors.

In summary, we have the following equivalent propositions.

- There exists a connected b-detachment exist.
- There exists a spanning tree of Γ in which all the edges have a different color.
- There exist the bases of a graph matroid of Γ which are also the independent sets of a partition matroid of color assignment.(The *rank* of base of a graph matroid is $|W|-1$. Since we use at most one edge of each color in the spanning tree, these edges correspond to the independent sets of the partition matroid.)

Proposition 1.3. *There exist the bases of a graph matroid of* Γ *which are also the independent sets of a partition matroid of color assignment, if and only if* $\forall F \subseteq E, \Gamma(E)\backslash\Gamma(F)$ *have at most* $|F|+1$ *components.*

Proof. Let π be the *rank* function and $\mathscr{I}$ be the independent set of the partition matroid . Let ρ be the *rank* function and $\mathscr{F}$ be the independent set of the graph matroid of Γ. From the theorem in 7th lecture, we get

$$\max\{|I|\big|I \in \mathscr{I} \cap \mathscr{F}\} = \min\{\pi(X) + \rho(E-X)\big|X \subseteq E\} \tag{3}$$

Let us consider the bipartite graph in which the vertices are E and $\Gamma(E)$. Figure 5 shows the example of the bipartite graph. In figure 5, whichever we choose Y as a gray region and a solid line region, we have $\pi(Y) = |F|$. This equals the *rank* of partition matroid of $\Gamma(F)$, that is, $\pi(\Gamma(F)) = |F|$. We can therefore assume that Y is equal to the $\Gamma(F)$. Then we have

$$\max\{|I|\big|I \in \mathscr{I} \cap \mathscr{F}\} = \min\{|F| + \rho(\Gamma(E)\backslash\Gamma(F))\big|F \subseteq E\} \tag{4}$$

Let $I \in \mathscr{I} \cap \mathscr{F}$ be the base of the graph matroid of Γ and the independent set of the partition matroind, which satisfies $|I| = |W|-1$. Therefore

$$\forall F \subseteq E, \rho(\Gamma(E)\backslash\Gamma(F)) + |F| \geq |W|-1. \tag{5}$$

Let $c(X)$ represent the component of X. We have

$$\rho(\Gamma(E)\backslash\Gamma(F)) = |W| - c(\Gamma(E)\backslash\Gamma(F)). \tag{6}$$

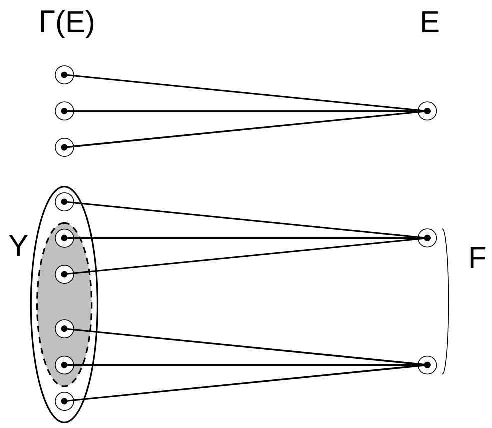

Figure.5 the example of the bipartite graph in which E and $\Gamma(E)$ are vertices.

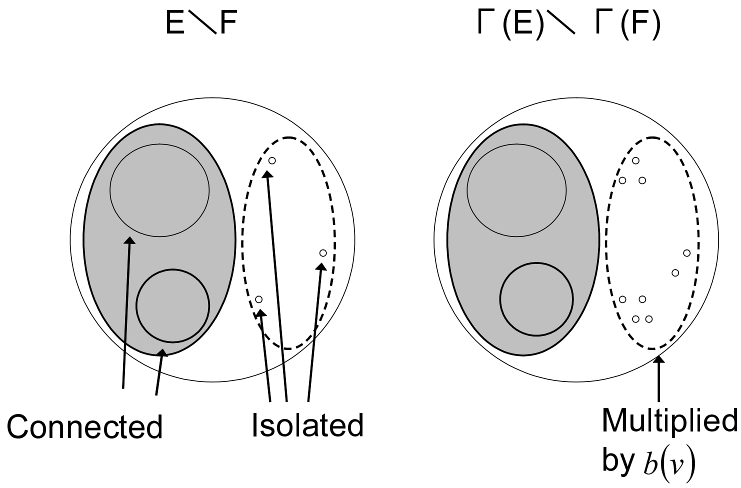

Figure.6 Example ofcomponents of $\Gamma(E)\backslash\Gamma(F)$

which follows

$$\forall F \subseteq E, c(\Gamma(E)\backslash\Gamma(F)) \leq |F| + 1. \tag{7}$$

In summary, there exist a connected b-detachment if and only if $\forall F \subseteq E, \Gamma(E)\backslash\Gamma(F)$ has at most $|F| + 1$ components □

To proof the Nash-Williams theorem, we need to show that $\forall F \subseteq E, c(\Gamma(E)\backslash\Gamma(F)) \leq |F| + 1$ if and only if $\forall X \subseteq V, b(X) + c(G\backslash X) \leq e(X) + 1$

Let H_F and I_F denote

$$H_F : \quad \Gamma(E)\backslash\Gamma(F)$$
$$I_F : \quad \text{The set of isolated vertices of } G\backslash F.$$

The isolated vertices of $c(G\backslash F)$ appeares twice in $c(G\backslash F)$ and the number of a component parts which are not isolated vertcies is not changed. Therefore we have

$$c(H_F) = c(G\backslash F) + b(I_F) - |I_F|. \tag{8}$$

We also have

$$c(\Gamma(E)\backslash\Gamma(F)) \leq |F| + 1, \forall F \subseteq E. \tag{9}$$

Using the equatoins, we have

$$c(G\backslash F) - |F| + b(I_F) - |I_F| \leq 1, \forall F \subseteq E. \tag{10}$$

Let F' be a set of edges which make I_F the isolated points, that is

$$I_{F'} = I_F F' \subseteq F. \tag{11}$$

Using F', (10) is equivalent to

$$c(G\backslash F') - |F'| + b(I_F) - |I_F| \leq 1, \forall F \subseteq E. \tag{12}$$

Let $e(X)$ denote the number of edge connecting to X. Since $c(G\backslash F') - |I_F| = c(G\backslash I_{F'}) = c(G\backslash I_F)$ and $|F'| :=e(I_{F'})$, we have

$$c(G\backslash I_F) - e(I_F) + b(I_F) \leq 1, \forall F \subseteq E. \tag{13}$$

Since we can assume I_F for $\forall F$, we define I_F as X,

$$c(G\backslash X) - e(X) + b(X) \leq 1, \forall X \subseteq V \tag{14}$$

□

2 Generalized Nash-Williams Theorem

We have the following theorem as a generalized version of Nash-Williams theorem.

Theorem 2.1. *If there exists a connected b-detachment of G then there exist a connected b-detachment of G, in which the degree sequence is equivalent to any degree sequences $\sigma(v)$, where a degree sequence $\sigma(v)$ satisfies $\sigma(v) = (\sigma_1(v) \dots \sigma_k(v))$, $k = b(v)$, $\sum_{i=1}^{k} \sigma_i(v) = d(v)$.*

Proof. For Γ, we assume that the connected spanning graph Q which uses each color just once and $\sum_{s \in W} |d_Q(s) - \sigma(s)|$ is minimum.

Suppose that the following equation does not satisfied.

$$d_Q(s) = \sigma(s), \forall s \in W. \tag{15}$$

Then, for some $v \in V$, there exist copies of $v, s, t \in W$ which satisfies

$$d_Q(s) > \sigma(s) \tag{16}$$
$$d_Q(s) < \sigma(t). \tag{17}$$

Let

$$e \in Q : \text{forbidden edge} \iff s \text{ and } t \text{ are not connected in } Q\backslash e.$$

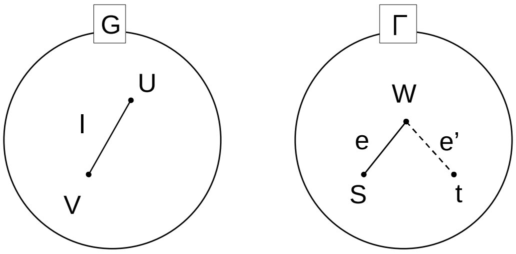

Figure.7 An example of exchange of edges

Let w be a copy of u. Then s connects at most one forbidden edge. Choose a edge e connecting s which is not forbidden edge. Remove e from Q and add e' to Q which is not in Q. Then $Q \backslash \{e\} \cap \{e'\}$ is connected. Now, since $d_Q(s)$ is decrease by one and $d_Q(t)$ is increased by one, $\sum_{s \in W} |d_Q(s) - \sigma(s)|$ is decreased by two. This contradicts the minimality of $\sum_{s \in W} |d_Q(s) - \sigma(s)|$.
Therefore $\sum_{s \in W} |d_Q(s) - \sigma(s)|$ is equal to 0. □

3 Extension of Euler's Theorem

Let G be a connected euler graph. Since G is an euler graph, we can consider b-detachment which satisfies the following equation:

$$b(v) = \frac{d(v)}{2} \ (v \in V) \tag{18}$$

and each node has a degree as follows:

$$\sigma(v) = (2, 2, \ldots, 2) \tag{19}$$

Apllying Nash-Williams's theorem to G, we have

$$b(X) + c(G \backslash X) \leq e(X) + 1 \tag{20}$$

We also have

$$b(X) = \sum_{v \in X} \frac{d(v)}{2} = e(X) - \frac{d(X, E \backslash X)}{2}. \tag{21}$$

Using the above equations we have

$$c(G \backslash X) \leq \frac{1}{2} d(X, E \backslash X) + 1. \tag{22}$$

Since we have equality only when b-detachment is a spanning tree. Since the b-detachmnet inlucde a cycle, we have

$$c(G \backslash X) < \frac{1}{2} d(X, E \backslash X) + 1, \tag{23}$$

which follows

$$2c(G \backslash X) \leq d(X, E \backslash X) \tag{24}$$

since G is an Euler graph and $d(X, E \backslash X)$ is even.