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Lecture 11 (Submodular Function 2)

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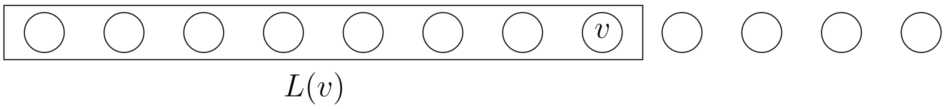
1 Greedy Algorithm

Let L be a linear order of a set V. That is, elements of V stand in a line as follows. We define $L(v)$ as a set which consists of elements from the first element to the element v.



We define y as follows :

$$y(v) = f(L(v)) - f(L(v) \setminus \{v\}) \ (v \in V).$$

Then, we have the following theorem.

Theorem 1. *y is a endpoint of $B(f)$.*

Proof. First, we show that $y(X) \le f(X)$ holds for all $X \subseteq V$:

$$\begin{aligned} y(X) &= \sum_{v \in X} (f(L(v)) - f(L(v) \setminus \{v\})) \\ &\le \sum_{v \in X} (f(X \cap L(v)) - f(X \cap L(v) \setminus \{v\})) \\ &= f(X). \end{aligned}$$

The inequality is due to the submodular property. If $v \in X$, the inequality

$$f(L(v)) + f(X \cap L(v) \setminus \{v\}) \le f(X \cap L(v)) + f(L(v) \setminus \{v\})$$

holds because the union of $X \cap L(v)$ and $L(v) \setminus \{v\}$ is $L(v)$ and the intersection is $X \cap L(v) \setminus \{v\}$. Therefore, $y \in P(f)$.

We also have $y \in B(f)$ because $y(V) = f(V)$. Actually,

$$\forall v \in V, \ y(L(v)) = f(L(v))$$

holds and thus there are n independent equalities. From this, we can conclude that y is a endpoint. □

As a corollary of this theorem, we obtain the following one.

Corollary 2. *The base polytope $B(f)$ is not empty.*

This means that the core of the convex game is not empty.

2 Max-Min Theorem

Theorem 3. $\min_{X \subset V} f(X) = \max\{z(V) \mid z \in P(f),\ z \leq 0\}$

Proof. Since $z \leq 0$, which means that z takes 0 for all elements, $z(V) \leq z(X)$ holds. On the other hand, $z(X) \leq f(X)$ holds because $z \in P(f)$. Therefore, $z(V) \leq f(X)$ holds and the left value is greater than the right value.

Next, we define f° as:

$$f^\circ(X) = \min\{f(Z) \mid Z \subset X\}.$$

f° is submodular because

$$\begin{aligned} f^\circ(X) + f^\circ(Y) &= f(X') + f(Y') \\ &\geq f(X' \cup Y') + f(X' \cap Y') \\ &\geq f^\circ(X \cup Y) + f^\circ(X \cap Y) \end{aligned}$$

from submodularity of f and the definition of f° where

$$\begin{aligned} f^\circ(X) = f(X'),\ X' \subset X \\ f^\circ(Y) = f(Y'),\ Y' \subset Y. \end{aligned}$$

Then, we have $B(f^\circ) \subseteq P(f^\circ) \subset P(f)$ because $f^\circ(X) \leq f(X)$ and $\forall X \subset V,\ f^\circ(X) \leq 0$ because $f^\circ(X) \leq f(\{\phi\}) = 0$. Since we know that $B(f^\circ)$ is not empty, if we take an arbitrary element z from $B(f^\circ)$, we have

$$\forall z \in B(f^\circ),\ \forall v \in V,\ z(v) \leq 0.$$

Now, z satisfies $z \in P(f),\ z \leq 0$ and is in $B(f^\circ)$. Thus,

$$z(V) = f^\circ(V) = \min\{f(Z) \mid Z \subseteq V\}$$

holds and the equality is proven. ◻

We can also formalize this theorem as follows. We define x^- as:

$$x^-(v) = \min\{0, x(v)\}.$$

Then,

$$x \in B(f) \Longrightarrow x^- \in P(f),\ x^- \leq 0$$

is clear. Conversely, we can see

$$z \in P(f),\ z \leq 0 \Longrightarrow \exists x \in B(f),\ x \geq z.$$

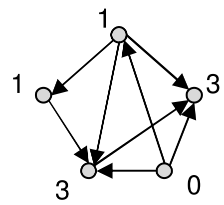
Thus, we have the following equalities.

$$\max\{z(V) \mid z \in P(f),\ z \leq 0\} = \max\{x^-(V) \mid x \in B(f)\} = \min_{X \subseteq V} f(X)$$

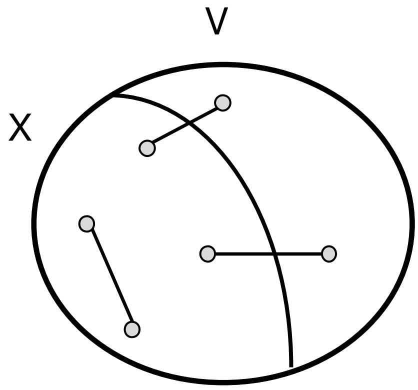
The argument above develop into the algorithms for minimization of submodular functions, but we will not discuss this here and give some references [1, 2].

3 Graph Orientation

An orientation of a graph determines an indegree of each vertex of the graph.



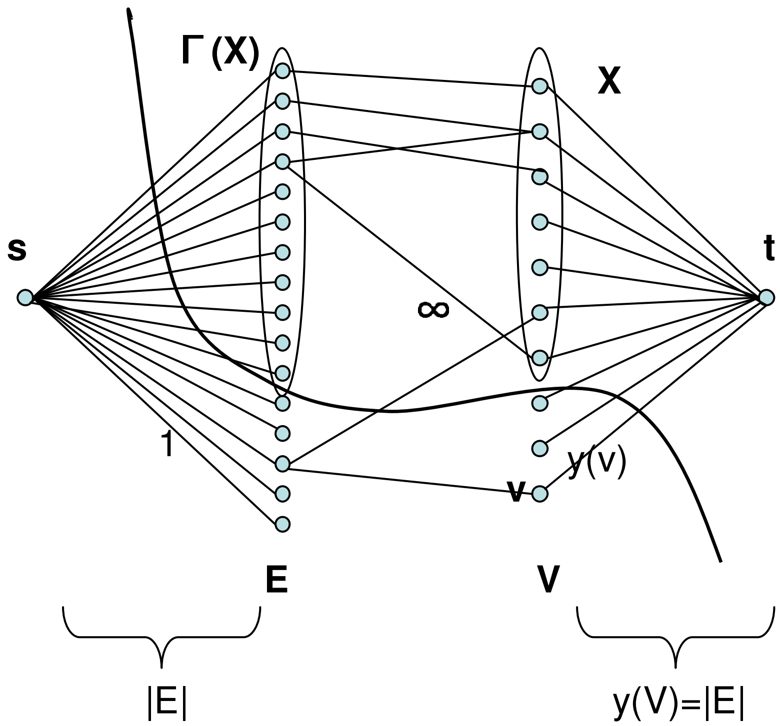
We consider the condition f



Denote the number of edges connecting to vertex subset $X \subseteq V$ by $e(X)$. It is obvious the condition

$$\begin{cases} \forall X \subseteq V, & y(X) \leq e(X) \\ & y(V) = |E| \end{cases}$$

is a necessary condition of achievement. We prove the condition is also a sufficient condition.



Construct a bipartite graph with vertex classes E and V, where $e \in E$ and $v \in V$ are adjacent if and only if v is an endpoint of e. Let an edge set connecting to $X \subseteq V$ be $\Gamma(X)$, then the condition for achievement can be restated into the condition

$$\forall X \subseteq V, \quad |\Gamma(X)| - y(X) \geq 0.$$

We prove if this condition is satisfied there exists the orientation.

Construct a network by adding two vertexes s and t to the bipartite graph, connecting s and $e \in E$ by an edge with capacity 1, and connecting $v \in V$ and t by an edge with capacity $y(v)$. We determine capacities of each original edge in the bipartite graph to be ∞ or 0 according to its orientation. For arbitrary X we consider a cut in the above figure, then its capacity is $|\Gamma(X)| + y(V \backslash X) = |\Gamma(X)| + |E| - y(X)$ neglecting capacities of the original edges in the bipartite graph. As the value is more than or equal to $|E|$, from the maximum-flow minimum-cut theorem we can conclude its maximum-flow equals to its minimum-cut $|E|$. This indicates that we can set an indegree of each $v \in V$ to be $y(v)$ by an appropriate orientation.

Theorem 4. *The set function e is a submodular function.*

Proof. Denote by $m(X,Y)$ the number of edges connecting X and Y. Then, it follows that

$$\begin{aligned} e(X \cup Y) + e(X \cap Y) &= e(X) + e(Y \backslash X) - m(X, Y \backslash X) + e(X \cap Y) \\ &= e(X) + e(Y \backslash X) - (m(X \backslash Y, Y \backslash X) + m(X \cap Y, Y \backslash X)) + e(X \cap Y) \\ &= e(X) + (e(Y \backslash X) + e(Y \cap X) - m(Y \cap X, Y \backslash X)) - m(X \backslash Y, Y \backslash X) \\ &= e(X) + e(Y) - m(X \backslash Y, Y \backslash X) \\ &\leq e(X) + e(Y). \end{aligned}$$

□

Reference

[1] A. Schrijver: A Combinatorial Algorithm Minimizing Submodular Functions in Strongly Polynomial Time. Journal of Combinatorial Theory, Series B 80 (2000), pp. 346-355.

[2] S. Iwata, L. Fleischer, S. Fujishige: A combinatorial strongly polynomial algorithm for minimizing submodular functions. Journal of the ACM, 48 (2001), pp. 761-777.

[3] L.S. Shapley: Cones of convex games. International Journal of Game Theory, 1 (1971), pp. 11-26.