Nash-Williams' theorem

Let $G = (V, E)$ be a graph. We denote the degree of $v \in V$ as $d(v)$, and define $b : V \to \mathbb{Z}$, which satisfies $0 \leq b(v) \leq d(v)$, $v \in V$.

**Theorem 1 (Nash Williams).** Following two conditions are equivalent.

1. There exists $b$-detachment of $G$.
2. $\forall X \subseteq V, b(X) \leq e(X) - c(G \setminus X) + 1$, when $b(X) = \sum_{v \in X} b(v)$, let $c(X)$ be the number of edges of $X$, and $c(G \setminus X)$ be the number of connected components of the graph which is derived by removing $X$ and all edges connected to $X$ from $G$.

**Proof.**

- $1 \to 2$
  This is proved in previous lecture.
- $2 \to 1$
  We define $f(X) = e(X) - c(G \setminus X) + 1$, then $f(\emptyset) = 0$, and $f(V) = |E| + 1$. Then,

**Lemma 1.** $f$ is submodular function.

**Proof.** Let $X, Y, Z \subseteq V$, and $u, v \in V$ be $X = Z \cup \{u\}$, $Y = Z \cup \{v\}$, $v \in V$, and $e'(v)$ be the number of outgoing edges of $u$ toward any vertices of $Z$. Let $m$ be the number of connected components which are connected to $u$, but not connected to $Z$ and $b$, and $n$ be the number of connected components which are connected to $v$, but not connected to $Z$ and $u$.

1. There are no edges between $u$ and $v$, and no vertices which are connected to both $u$ and $v$.

![Diagram](image1.png) Fig. 1 There are no edges between $u$ and $v$, and no vertices which are connected to both $u$ and $v.
\[ f(X) + f(Y) = e(Z) + e'(u) - c(G \setminus (Z \cup \{u\})) + 1, \]
\[ + e(Z) + e'(v) - c(G \setminus (Z \cup \{v\})) + 1, \]
\[ = 2e(Z) + e'(u) + e'(v) - n - m, \]
\[ f(X \cup Y) + f(X \cup Y) = e(Z) + e'(u) + e'(v) - c(G \setminus (Z \cup \{u\} \cup \{v\})) + 1, \]
\[ + e(Z) - c(G \setminus Z) + 1, \]
\[ = 2e(Z) + e'(u) + e'(v) - n - m. \]

then
\[ (f(X) + f(Y)) - f(X \cup Y) + f(X \cup Y) = 0. \]

2. There are no edges between \( u \) and \( v \), and there exist \( k \geq 1 \) connected components which are connected to both \( u \) and \( v \).

![Diagram](image)

Fig. 2 There are no edges between \( u \) and \( v \), and there exist \( k \geq 1 \) connected components which are connected to both \( u \) and \( v \).

\[ f(X) + f(Y) = e(Z) + e'(u) - c(G \setminus (Z \cup \{u\})) + 1, \]
\[ e(Z) + e'(v) - c(G \setminus (Z \cup \{v\})) + 1, \]
\[ = 2e(Z) + e'(u) + e'(v) - n - m. \]
\[ f(X \cup Y) + f(X \cup Y) = e(Z) + e'(u) + e'(v) - c(G \setminus (Z \cup \{u\} \cup \{v\})) + 1, \]
\[ e(Z) - c(G \setminus Z) + 1, \]
\[ = 2e(Z) + e'(u) + e'(v) - n - m - k + 1. \]

then
\[ (f(X) + f(Y)) - f(X \cup Y) + f(X \cup Y) \geq k - 1. \]

3. There exist a edge between \( u \) and \( v \), and \( k \geq 0 \) connected components which are connected both \( u \) and \( v \).
There exist an edge between \( u \) and \( v \), and \( k (\geq 0) \) connected components which are connected both \( u \) and \( v \).

\[
\begin{align*}
  f(X) + f(Y) &= e(z) + e'(u) + 1 - c(G \setminus (Z \cup \{u\}) + 1, \\
 e(Z) + e'(v) + 1 - c(G \setminus (Z \cup \{v\})) + 1, \\
  &= 2e(Z) + e'(u) + e'(v) - n - m + 2.
\end{align*}
\]

\[
\begin{align*}
  f(X \cup Y) + f(X \cup Y) &= e(z) + e'(u) + e'(v) + 1 - c(G \setminus (Z \cup \{u\} \cup \{v\}) + 1, \\
  e(Z) - c(G \setminus Z) + 1, \\
  &= 2e(Z) + e'(u) + e'(v) - n - m - k + 2.
\end{align*}
\]

Then,

\[
(f(X) + f(Y)) - f(X \cup Y) + f(X \cup Y) \geq k.
\]

This implies that 2 of theorem 1 means \( b \in P(f) (P:\text{submodular polytope}) \), then we proof

\[
b \in P(f) \rightarrow \exists b\text{-detachment}.
\]

\[
\begin{align*}
  f(\emptyset) &= 0, \\
  f(V) &= |E| + 1, \\
  \forall X \not\subseteq V, f(X) \leq e(X) : f(X) = e(X) - c(G \setminus X) + 1.
\end{align*}
\]

\[
\exists h \in B(f), h \geq b (B:\text{base polytope}), \text{because } b \in P(f). \text{ Choose randomly } r \in V. \ y(V) = |E|, \text{ when } y = h - X_r.
\]

Then,

\[
\begin{align*}
  y(X) &\leq h(X) \leq f(X) \leq e(X), \forall X \not\subseteq V, \\
  y(V) &= |E| = e(V).
\end{align*}
\]

This implies

\[
y \in B(e).
\]

There exists at least one orientation whose indegree are equal to \( y \).

Let \( X \) be a set of vertices which is reachable from \( r \).
There are no outgoing edges from $X$.

$$y(X) = e(X) \leq h(X) - 1 \leq f(X).$$

This implies $X = V$. There exists a directed minimum spanning tree $T$ whose root is $r$, because all vertices in $V$ are reachable from $r$.

Each $v \in V$ has $h(v) - 1$ edges in $E \setminus T$ whose end is $v$. This implies that there exists $b$-detachment of $G$. 

$\Box$
Fig. 7 Detached vertices are end points, and they are reachable from $r$. 