Discrete Methods in Informatics January 30, 2005

Lecture 12 (Submodular function 3)

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Nash-Williams' theorem

Let $G = (V, E)$ be a graph. We denote the degree of $v \in V$ as $d(v)$, and define $b : V \to \mathbb{Z}$, which satisfies $0 \le b(v) \le d(v)$, $v \in V$.

Theorem 1 (Nash Williams). *Following two conditions are equivalent.*

1. *There exists b-detachment of G.*
2. *$\forall X \subseteq V$, $b(X) \le e(X) - c(G \setminus X) + 1$, when $b(X) = \sum_{v \in X} b(v)$, let $e(X)$ be the number of edges of X, and $c(G \setminus X)$ be the number of connected components of the graph which is derived by removing X and all edges connected to X from G.*

Proof.

- $1 \to 2$
  This is proved in previous lecture.
- $2 \to 1$
  We define $f(X) = e(X) - c(G \setminus X) + 1$, then $f(\emptyset) = 0$, and $f(V) = |E| + 1$. Then,

  Lemma 1. *f is submodular function.*

  Proof. Let $X, Y, Z \subseteq V$, and $u, v \in V$ be $X = Z \cup \{u\}$, $Y = Z \cup \{v\}$, $v \in V$, and $e'(v)$ be the number of outgoing edges of u toward any vertices of Z. Let m be the number of connected components which are connected to u, but not connected to Z and b, and n be the number of connected components which are connected to v, but not connected to Z and u.

  1. There are no edges between u and v, and no vertices which are connected to both u and v.

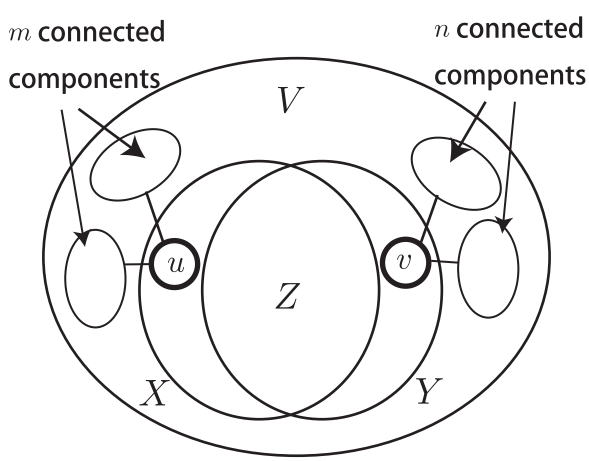

Fig. 1 There are no edges between u and v, and no vertices which are connected to both u and v.

$$
\begin{aligned}
f(X)+f(Y) &= e(Z)+e'(u)-c(G\setminus(Z\cup\{u\})+1,\\
&\quad +e(Z)+e'(v)-c(G\setminus(Z\cup\{v\}))+1,\\
&= 2e(Z)+e'(u)+e'(v)-n-m.\\
f(X\cup Y)+f(X\cup Y) &= e(Z)+e'(u)+e'(v)-c(G\setminus(Z\cup\{u\}\cup\{v\})+1,\\
&\quad +e(Z)-c(G\setminus Z)+1,\\
&= 2e(Z)+e'(u)+e'(v)-n-m.
\end{aligned}
$$

then

$$(f(X)+f(Y))-f(X\cup Y)+f(X\cup Y)=0.$$

2. There are no edges between u and v, and there exist $k(\geq 1)$ connected components which are connected to both u and v.

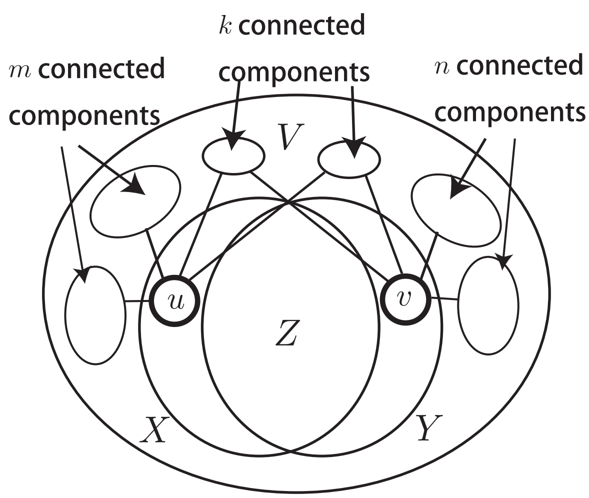

Fig. 2 There are no edges between u and v, and there exist $k(\geq 1)$ connected components which are connected to both u and v.

$$
\begin{aligned}
f(X)+f(Y) &= e(z)+e'(u)-c(G\setminus(Z\cup\{u\})+1,\\
&\quad e(Z)+e'(v)-c(G\setminus(Z\cup\{v\}))+1,\\
&= 2e(Z)+e'(u)+e'(v)-n-m.\\
f(X\cup Y)+f(X\cup Y) &= e(Z)+e'(u)+e'(v)-c(G\setminus(Z\cup\{u\}\cup\{v\})+1,\\
&\quad e(Z)-c(G\setminus Z)+1,\\
&= 2e(Z)+e'(u)+e'(v)-n-m-k+1.
\end{aligned}
$$

then

$$(f(X)+f(Y))-f(X\cup Y)+f(X\cup Y)\geq k-1.$$

3. There exist a edge between u and v, and $k(\geq 0)$ connected components which are connected both u and v.

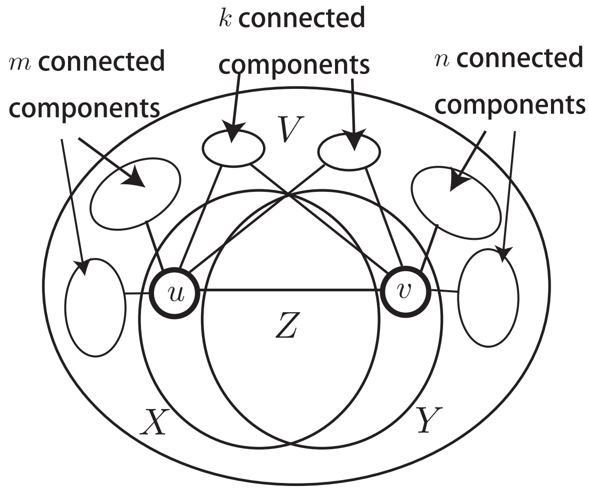

Fig. 3 There exist a edge between u and v, and $k(\geq 0)$ connected components which are connected both u and v.

$$\begin{aligned} f(X)+f(Y) &= e(z)+e'(u)+1-c(G\setminus(Z\cup\{u\})+1, \\ & \quad e(Z)+e'(v)+1-c(G\setminus(Z\cup\{v\}))+1, \\ &= 2e(Z)+e'(u)+e'(v)-n-m+2. \\ f(X\cup Y)+f(X\cup Y) &= e(Z)+e'(u)+e'(v)+1-c(G\setminus(Z\cup\{u\}\cup\{v\})+1, \\ & \quad e(Z)-c(G\setminus Z)+1, \\ &= 2e(Z)+e'(u)+e'(v)-n-m-k+2. \end{aligned}$$

Then,

$$(f(X)+f(Y))-f(X\cup Y)+f(X\cup Y)\geq k.$$

□

This implies that 2 of theorem 1 means $b\in P(f)$(P:submodular polytope), then we proof

$$b\in P(f)\rightarrow \exists b\text{-detachment}.$$

$$\begin{aligned} & f(\emptyset)=0, \\ & f(V)=|E|+1, \\ & \forall X\subsetneq V, f(X)\leq e(X) \quad \because f(X)=e(X)-c(G\setminus X)+1. \end{aligned}$$

$\exists h\in B(f)$, $h\geq b$ (B:base polytope), because $b\in P(f)$. Choose randomly $r\in V$. $y(V)=|E|$, when $y=h-X_r$.
Then,

$$\begin{aligned} & y(X)\leq h(X)\leq f(X)\leq e(X), \forall X\subsetneq V, \\ & y(V)=|E|=e(V). \end{aligned}$$

This implies

$$y\in B(e).$$

There exists at least one orientation whose indegree are equal to y.
Let X be a set of vertices which is reachable from r.

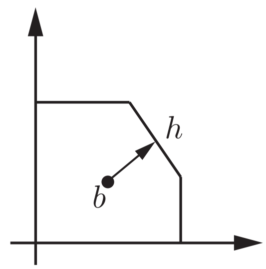

Fig. 4 $\exists h \in B(f),\, h \geq b$

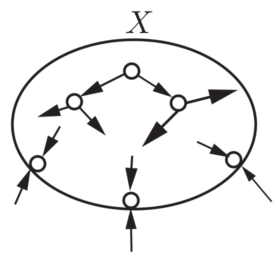

Fig. 5 There are no outgoing edges from X.

$$y(X) = e(X) \leq h(X) - 1 \leq f(X).$$

This implies $X = V$. There exists a directed minimum spanning tree T whose root is r, because all vertices in V are reachable from r.

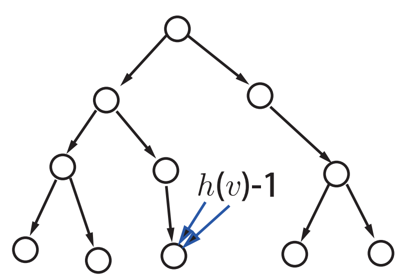

Fig. 6 Each v has $h(v) - 1$ edges in $E \setminus T$ whose end is v.

Each $v \in V$ has $h(v) - 1 (\geq b(v) - 1)$ edges in $E \setminus T$ whose end are v. This implies that there exists b-detachment of G.

□

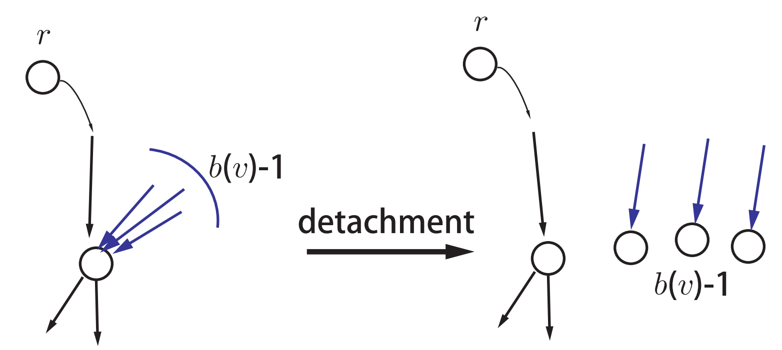

Fig. 7 Detached vertices are end points, and they are reachable from r.