SIAM J. CONTROL OPTIM.
Vol. 55, No. 3, pp. 2134–2150

ON THE KRONECKER CANONICAL FORM OF SINGULAR MIXED MATRIX PENCILS*

SATORU IWATA† AND MIZUYO TAKAMATSU‡

Abstract. Consider a linear time-invariant dynamical system that can be described as $F\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$, where A, B, and F are mixed matrices, i.e., matrices having two kinds of nonzero coefficients: fixed constants that account for conservation laws and independent parameters that represent physical characteristics. The controllable subspace of the system is closely related to the Kronecker canonical form of the mixed matrix pencil $\left(A - sF \mid B\right)$. Under a physically meaningful assumption justified by the dimensional analysis, we provide a combinatorial characterization of the sums of the minimal row/column indices of the Kronecker canonical form of mixed matrix pencils. The characterization leads to a matroid-theoretic algorithm for efficiently computing the dimension of the controllable subspace for the system with nonsingular F.

Key words. structural controllability, controllable subspace, Kronecker canonical form, matrix pencil, matroid theory, mixed matrix

AMS subject classifications. 93B05, 93B25, 15A22, 05C50

DOI. 10.1137/16M1057516

1. Introduction.

A *matrix pencil* is a polynomial matrix in which the degree of each entry is at most one. Each matrix pencil is known to be strictly equivalent to its *Kronecker canonical form*. The Kronecker canonical form plays an important role in various fields such as systems control [6, 38] and differential-algebraic equations [2, 12, 20, 34]. Several algorithms are designed for numerically stable computation of the Kronecker canonical form [1, 7, 8, 18, 40].

An alternative method for the Kronecker canonical form is based on the so-called structural approach, which extracts a zero/nonzero pattern of each coefficient in the matrix pencil, ignoring the numerical values. The structural approach enables us to compute the Kronecker canonical form of regular matrix pencils efficiently by exploiting graph-algorithmic techniques under the genericity assumption that all the nonzero coefficients are independent parameters which do not cause any numerical cancellation. A recent work [15] has extended the structural approach to deal with singular matrix pencils.

The structural approach dates back to the 1970s. In 1974, Lin [21] introduced the notion of *structural controllability*, which leads to the development of the structural approach in control theory [11, 13, 24, 37]. The structural approach has also been developed in theory of differential-algebraic equations [32, 33]. In the present day, many modeling and simulation tools for dynamical systems including Dymola [39] adopt algorithms based on the structural approach.

An advantage of the structural approach is that it is supported by efficient combinatorial algorithms that are free from errors in numerical computation. On the other

*Received by the editors January 20, 2016; accepted for publication (in revised form) February 8, 2017; published electronically June 29, 2017.
http://www.siam.org/journals/sicon/55-3/M105751.html
Funding: This research is supported by JST CREST grant JPMJCR14D2. The second author's research is supported in part by JSPS KAKENHI grant 25730009.
†Department of Mathematical Informatics, The University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113-8656, Japan (iwata@mist.i.u-tokyo.ac.jp).
‡Department of Information and System Engineering, Chuo University, Kasuga 1-13-27, Bunkyo-ku, Tokyo 112-8551, Japan (takamatsu@ise.chuo-u.ac.jp).

hand, however, the genericity assumption is often invalid when we set up a faithful model of a physical system. This is partly because structural equations such as the conservation laws can be described with specific numbers. This natural observation led Murota and Iri [29] to introduce the notion of a *mixed matrix*, which is a constant matrix that consists of two kinds of numbers as follows.

Accurate Numbers (Fixed Constants) Numbers that account for conservation laws are precise in values. These numbers should be treated numerically.

Inaccurate Numbers (Independent Parameters) Numbers that represent physical characteristics are not precise in values. These numbers should be treated combinatorially as nonzero parameters without reference to their nominal values. Since each such nonzero entry often comes from a single physical device, the parameters are assumed to be independent.

The polynomial matrix version of a mixed matrix is called a *mixed polynomial matrix*. To be more specific, a mixed polynomial matrix is a polynomial matrix $D(s) = Q(s) + T(s)$ such that the nonzero entries in the coefficient matrices of $Q(s)$ are fixed constants and those of $T(s)$ are independent parameters.

The concept of mixed polynomial matrices may be too broad as a mathematical tool for describing dynamical systems in practice. Taking the consistency of physical dimensions in structural equations into account, Murota [22] introduced a class of mixed polynomial matrices that satisfy the following condition.

(DC) Every nonvanishing subdeterminant of $Q(s)$ is a monomial in s.

This subclass of mixed polynomial matrices has played an important role in the matroid-theoretic structural approach to dynamical systems [23, 24, 25, 26, 31].

The Kronecker canonical form consists of nilpotent blocks, rectangular blocks, and the other square blocks. The size of each rectangular block is called the *minimal row/column indices*. Under the genericity assumption, [15] provides a combinatorial characterization of the sizes of the nilpotent blocks as well as the sums of the minimal row/column indices. The results on the nilpotent blocks have been successfully extended to the framework of *mixed matrix pencils*, i.e., mixed polynomial matrices with degree at most one, without imposing the assumption (DC) on dimensional consistency [16]. In this paper, we extend the characterization on the sums of the minimal row/column indices to the framework of mixed matrix pencils satisfying (DC).

The minimal row/column indices have been characterized in terms of the Wong sequences by Berger and Trenn [3, 4]. Their method can also deal with mixed matrix pencils of moderate size with the aid of symbolic computation. The computational cost, however, can grow explosively when the size increases. In order to overcome this drawback, we aim at developing a method that does not rely on symbolic computation. In fact, our characterization leads to an efficient matroid-theoretic algorithm that consists of graph manipulation and matrix computation over the accurate numbers.

In control theory, the minimal row/column indices are also referred to as the left/right Kronecker indices [19, 41]. For a linear time-invariant dynamical system $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$, the minimal column indices of the matrix pencil $D(s) = \big(sI - A \mid B\big)$ provides the so-called controllability indices [14, 19, 35, 42, 43], and the sum of the minimal column indices corresponds to the dimension of the controllable subspace. The definition of the controllable subspace is generalized to a linear time-invariant dynamical system in a descriptor form $F\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$. The study of the augmented Wong sequences has been developed to characterize fundamental subspaces including the controllable subspace [2].

For the cases when A, B, and F are mixed matrices, Murota [23] has presented a matroid-theoretic algorithm for testing the controllability of this system.

This algorithm, however, does not provide the dimension of the controllable subspace. Our characterization of the sum of the minimal column indices leads to an algorithm for computing the dimension of the controllable subspace under the assumption that F is nonsingular.

The descriptor form $F\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t)+B\boldsymbol{u}(t)$ with nonsingular F is mathematically equivalent to $\dot{\boldsymbol{x}}(t) = F^{-1}A\boldsymbol{x}(t) + F^{-1}B\boldsymbol{u}(t)$. We should remark, however, that $F^{-1}A$ and $F^{-1}B$ are no longer mixed matrices, and hence we cannot apply our algorithm to the latter formulation. This is the reason why we should deal with the descriptor form directly rather than reducing it to the standard form.

In the derivation of our result, we have two difficulties to overcome. In mixed matrix theory, a problem for a mixed matrix pencil is generally reduced to that for a certain *layered mixed matrix pencil*, but this straightforward approach does not work well for the minimal column indices as discussed in [16, section 8]. This is the first difficulty, which is resolved by Theorem 4.3. The second one occurs in using the *combinatorial canonical form (CCF)* decomposition [30]. When we transform a mixed matrix pencil $D(s)$ into the CCF, the resulting matrix is not necessarily a matrix pencil. We resolve this problem by showing in section 6 that a part of the CCF, called the *horizontal tail*, remains to be a matrix pencil and has the same minimal column indices as $D(s)$.

The rest of this paper is organized as follows. In section 2, we recapitulate the Kronecker canonical form. Section 3 discusses which blocks are invariant under equivalence transformations with unimodular matrices. Sections 4 is devoted to mixed matrix pencils. After expounding the CCF in mixed matrix theory in section 5, we give a combinatorial characterization of the sums of the minimal row/column indices in section 6. Section 7 describes an application of our result to controllability analysis of dynamical systems. Finally, section 8 concludes this paper.

2. The Kronecker canonical form of matrix pencils. In this section, we discuss matrix pencils over an arbitrary field $\mathbf{F}$. Let $D(s) = sX + Y$ be an $m \times n$ matrix pencil with row index set R and column index set C. We denote by $D(s)[I, J]$ the submatrix of $D(s)$ determined by $I \subseteq R$ and $J \subseteq C$. A matrix pencil $D(s)$ is said to be *regular* if $D(s)$ is square and $\det D(s) \neq 0$ as a polynomial in s. It is *strictly regular* if both X and Y are nonsingular. The rank of $D(s)$ is the maximum size of its submatrix that is a regular matrix pencil. A matrix pencil $\bar{D}(s)$ is said to be *strictly equivalent* to $D(s)$ if there exists a pair of nonsingular constant matrices U and V such that $\bar{D}(s) = UD(s)V$.

For $\mu \geq 1$ and $\epsilon \geq 0$, we consider $\mu \times \mu$ matrix pencils K_μ, N_μ and an $\epsilon \times (\epsilon + 1)$ matrix pencil L_ϵ defined by

$$K_\mu = \begin{pmatrix} s & 1 & 0 & \cdots & 0 \\ 0 & s & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & s & 1 \\ 0 & \cdots & \cdots & 0 & s \end{pmatrix}, \quad N_\mu = \begin{pmatrix} 1 & s & 0 & \cdots & 0 \\ 0 & 1 & s & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & s \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \quad L_\epsilon = \begin{pmatrix} s & 1 & 0 & \cdots & 0 \\ 0 & s & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s & 1 \end{pmatrix}.$$

The transpose matrix of L_η is denoted by $L_\eta^\top$. Let us denote by block-diag$(D_1, \ldots, D_h)$ the block-diagonal matrix with diagonal blocks $D_1, \ldots, D_h$. A matrix pencil over $\mathbb{C}$ is known to be strictly equivalent to a block-diagonal form called the *Kronecker canonical form* [10, Chapter XII]. The corresponding statement applicable over an arbitrary field $\mathbf{F}$ is given as follows.

THEOREM 2.1. *A matrix pencil $D(s)$ over a field $\mathbf{F}$ is strictly equivalent to a block-diagonal form $\bar{D}(s)$ with*

$$\bar{D}(s) = \text{block-diag}(H_\nu, K_{\rho_1}, \ldots, K_{\rho_c}, N_{\mu_1}, \ldots, N_{\mu_d}, L_{\epsilon_1}, \ldots, L_{\epsilon_p}, L_{\eta_1}^\top, \ldots, L_{\eta_q}^\top),$$

where $c, d, p, q \geq 0$, $\rho_1 \geq \cdots \geq \rho_c \geq 1$, $\mu_1 \geq \cdots \geq \mu_d \geq 1$, $\epsilon_1 \geq \cdots \geq \epsilon_p \geq 0$, $\eta_1 \geq \cdots \geq \eta_q \geq 0$, and H_ν is a strictly regular matrix pencil of size ν. The numbers ν, c, d, p, q, $\rho_1, \ldots, \rho_c$, $\mu_1, \ldots, \mu_d$, $\epsilon_1, \ldots, \epsilon_p$, $\eta_1, \ldots, \eta_q$ are uniquely determined.

When $\mathbf{F} = \mathbb{C}$, the strictly regular part H_ν can further be brought into $sI_\nu + J_\nu$ with J_ν being the Jordan normal form. The resulting block-diagonal form is often called the Kronecker canonical form. In this paper, however, we use the same term for a matrix pencil over an arbitrary field $\mathbf{F}$ to mean the above block-diagonal form $\bar{D}(s)$. We remark that L_0 represents a 0×1 block. Having such a diagonal block corresponds to having a zero column. For example, block-diag(L_1, L_0) designates a matrix $\begin{pmatrix} s & 1 & 0 \end{pmatrix}$.

The matrices $N_{\mu_1}, \ldots, N_{\mu_d}$ are called the *nilpotent blocks*, and $\mu_1, \ldots, \mu_d$ are called the *indices of nilpotency*. The numbers $\epsilon_1, \ldots, \epsilon_p$ and $\eta_1, \ldots, \eta_q$ are the *minimal column indices* and *minimal row indices*, respectively. We collectively call $(\nu, \rho_1, \ldots, \rho_c, \mu_1, \ldots, \mu_d, \epsilon_1, \ldots, \epsilon_p, \eta_1, \ldots, \eta_q)$ the *structural indices* of $D(s)$. For the rank r of $D(s)$, it holds that

$$r = \nu + \sum_{i=1}^{c} \rho_i + \sum_{i=1}^{d} \mu_i + \sum_{i=1}^{p} \epsilon_i + \sum_{i=1}^{q} \eta_i, \quad p = n - r, \quad q = m - r. \tag{2.1}$$

We denote the degree of a polynomial $f(s)$ by $\deg f(s)$, where $\deg 0 = -\infty$ by convention. A rational function $f(s) = g(s)/h(s)$ with irreducible polynomials $g(s)$ and $h(s)$ is called a *Laurent polynomial* if $h(s)$ is monomial. The degree of $f(s)$ is defined by $\deg f(s) = \deg g(s) - \deg h(s)$. Note that $-\deg f(s)$ is equal to the relative degree of $f(s)$.

Let $B(s)$ be a Laurent polynomial matrix with row index set R and column index set C. For $k = 1, \ldots, \operatorname{rank} B(s)$, we denote

$$\delta_k(B(s)) = \max\{\deg \det B(s)[I, J] \mid |I| = |J| = k, I \subseteq R, J \subseteq C\},$$
$$\zeta_k(B(s)) = \min\{\operatorname{ord} \det B(s)[I, J] \mid |I| = |J| = k, I \subseteq R, J \subseteq C\},$$

where ord denotes the minimum degree of a nonzero term in a Laurent polynomial. We define $\delta_0(B(s)) = 0$ and $\zeta_0(B(s)) = 0$. By $\zeta_k(B(s)) = -\delta_k(B(1/s))$, we obtain

$$\delta_k(X + sY) = k + \delta_k\left(\frac{1}{s}X + Y\right) = k - \zeta_k(sX + Y). \tag{2.2}$$

For the indices of nilpotency of the Kronecker canonical form, it is known that

$$d = r - \max_{k \geq 0} \delta_k(D(s)), \quad \mu_i = \delta_{r-i}(D(s)) - \delta_{r-i+1}(D(s)) + 1 \tag{2.3}$$

hold [28, Theorem 5.1.8]. Since ρ_i $(i = 1, \ldots, c)$ of $D(s) = sX + Y$ coincides with μ'_i $(i = 1, \ldots, d')$ of $D'(s) = X + sY$, it follows from (2.2) and (2.3) that

$$c = d' = r - \max_{k \geq 0} \delta_k(D'(s)) = r + \min_{k \geq 0}(\zeta_k(D(s)) - k), \tag{2.4}$$

$$\rho_i = \mu'_i = \delta_{r-i}(D'(s)) - \delta_{r-i+1}(D'(s)) + 1 = \zeta_{r-i+1}(D(s)) - \zeta_{r-i}(D(s)). \tag{2.5}$$

In addition, it is shown in [15] that the following equation holds:

$$\nu + \sum_{i=1}^{p} \epsilon_i + \sum_{i=1}^{q} \eta_i = \delta_r(D(s)) - \zeta_r(D(s)). \tag{2.6}$$

Let $A(s)$ be an $m \times n$ polynomial matrix. The kth determinantal divisor $d_k(A(s))$ is defined to be the greatest common divisor of all the subdeterminants of order k:

$$d_k(A(s)) = \gcd\{\det A(s)[I, J] \mid |I| = |J| = k\} \quad (k = 0, 1, \ldots, \operatorname{rank} A(s)), \tag{2.7}$$

where $d_k(A(s))$ is chosen to be monic and $d_0(A(s)) = 1$ by convention. The following lemma characterizes the sum of the sizes of the H_ν block and K_{ρ_i} blocks.

Lemma 2.2. *For a matrix pencil $D(s)$ of rank r, we have $\nu + \sum_{i=1}^{c} \rho_i = \deg d_r(D(s))$.*

Proof. Let $\bar{D}(s)$ be the Kronecker canonical form of $D(s)$. We now have $d_\nu(H_\nu) = \det H_\nu$, $d_\rho(K_\rho) = s^\rho$, $d_\mu(N_\mu) = 1$, $d_\epsilon(L_\epsilon) = \gcd\{s^\epsilon, s^{\epsilon-1}, \ldots, 1\} = 1$, and $d_\eta(L_\eta^\top) = \gcd\{s^\eta, s^{\eta-1}, \ldots, 1\} = 1$. Hence $d_r(D(s)) = d_r(\bar{D}(s)) = s^{\rho_1+\cdots+\rho_c} \det H_\nu$ holds. Since H_ν is strictly regular, this implies $\deg d_r(D(s)) = \rho_1 + \cdots + \rho_c + \nu$. ∎

For an $m \times n$ matrix pencil $D(s) = sX + Y$, we consider a $(k+1)m \times kn$ matrix

$$\Psi_k(D) = \begin{pmatrix} X & O & \cdots & O \\ Y & X & \ddots & \vdots \\ O & Y & \ddots & O \\ \vdots & \ddots & \ddots & X \\ O & \cdots & O & Y \end{pmatrix}.$$

We denote the rank of $\Psi_k(D)$ by $\psi_k(D)$. The following equation shows a close relationship between $\psi_k(D)$ and $\epsilon_1, \ldots, \epsilon_p$ [15, Theorem 2.3]:

$$\psi_k(D) = rk + \sum_{i=1}^{p} \min\{k, \epsilon_i\}. \tag{2.8}$$

We generalize the definition of $\Psi_k(D)$ to that for a polynomial matrix as follows. Let $A(s) = \sum_{i=0}^{N} s^i A_i$ be an $m \times n$ polynomial matrix such that the maximum degree of entries is N. Given $A(s)$ and an integer l, we define a $(k+l)m \times kn$ matrix

$$\Psi_k^l(A) = \begin{array}{c} \\ R_0 \\ R_1 \\ \vdots \\ R_l \\ R_{l+1} \\ \vdots \\ R_{k+l-1} \end{array} \begin{array}{c} \begin{array}{cccc} C_0 & C_1 & \cdots & C_{k-1} \end{array} \\ \begin{pmatrix} A_0 & O & \cdots & O \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & A_1 & \ddots & O \\ A_l & \vdots & \ddots & A_0 \\ O & A_l & \ddots & A_1 \\ \vdots & \ddots & \ddots & \vdots \\ O & \cdots & O & A_l \end{pmatrix} \end{array}$$

with row index set $\tilde{R} = R_0 \cup R_1 \cup \cdots \cup R_{k+l-1}$ and column index set $\tilde{C} = C_0 \cup C_1 \cup \cdots \cup C_{k-1}$. We note that $A_h = O$ for $h > N$. The matrix $\Psi_k^1(A)$ coincides with

$\Psi_k(A_0 + sA_1)$, and the ranks of $\Psi_k^l(A)$ for $l \geq N$ attain the same value, which we denote by $\psi_k(A)$. The following lemma is a generalization of Corollary 2.4 in [15].

LEMMA 2.3. *If an $m \times n$ polynomial matrix $A(s)$ is of full-column rank, we have $\psi_k(A) = kn$ for each k.*

Proof. Let N denote the maximum degree of entries in $A(s)$. We assume that $\psi_k(A) \neq kn$, which implies that $\Psi_k^N(A)$ is not of full-column rank. Let $\boldsymbol{h}_j^l$ denote the lth column vector of $\Psi_k^N(A)[\tilde{R}, C_j]$. Then we have $\sum_{j=0}^{k-1}\sum_{l\in C_j} \lambda_j^l \boldsymbol{h}_j^l = \mathbf{0}$ for some λ_j^l such that scalars λ_j^l are not all zero.

Let C denote the column index set of $A(s)$. By the definition of $\Psi_k^N(A)$, a part of vector $\boldsymbol{h}_j^l$ indexed by R_i is equal to the lth vector of A_{i-j}, denoted by A_{i-j}^l, where we set $A_{i-j} = O$ if $i - j < 0$ or $i - j > N$. Hence it holds that

$$\sum_{j=0}^{k-1}\sum_{l\in C} \lambda_j^l A_{i-j}^l = \mathbf{0} \quad (i = 0, 1, \ldots, k + N - 1). \tag{2.9}$$

We denote the lth vector of $A(s)$ by $\boldsymbol{a}_l(s)$. Consider a linear combination $\boldsymbol{b}(s) = \sum_{l\in C}(\sum_{j=0}^{k-1} \lambda_j^l s^j)\boldsymbol{a}_l(s)$ of vectors in $A(s)$, where each coefficient is a polynomial in s. The coefficient of s^i in $\boldsymbol{b}(s)$ is expressed as $\sum_{l\in C}\sum_{j=0}^{k-1} \lambda_j^l A_{i-j}^l$, which is equal to $\mathbf{0}$ by (2.9). Hence $\boldsymbol{b}(s) = \mathbf{0}$ holds. This implies $A(s)$ is not of full-column rank. ∎

3. Invariance under unimodular equivalence transformations. A polynomial matrix is called *unimodular* if it is square and its determinant is a nonvanishing constant. For a polynomial matrix $A(s)$, $d_k(A(s))$ is invariant under unimodular equivalence transformations, that is, $d_k(A(s)) = d_k(A'(s))$ if $A'(s) = U(s)A(s)V(s)$ with unimodular matrices $U(s)$ and $V(s)$. The same applies to $\zeta_k(B(s))$ for a Laurent polynomial matrix $B(s)$.

For a matrix pencil $D(s)$, consider another matrix pencil $D'(s)$ obtained by $D'(s) = U(s)D(s)V(s)$ with some unimodular matrices $U(s)$ and $V(s)$. The structural indices of $D'(s)$ is denoted by $(\nu', \rho_1', \ldots, \rho_{c'}', \mu_1', \ldots, \mu_{d'}', \epsilon_1', \ldots, \epsilon_{p'}', \eta_1', \ldots, \eta_{q'}')$. We have $p = p'$ and $q = q'$ by (2.1), $c = c'$ by (2.4), and $\rho_i = \rho_i'$ $(i = 1, \ldots, c)$ by (2.5). Since $d_r(D(s)) = d_r(D'(s))$ with $r = \operatorname{rank} D(s) = \operatorname{rank} D'(s)$, $\nu = \nu'$ follows from Lemma 2.2.

Table 1 shows whether the size of each block is invariant or not under the following three kinds of transformations from $D(s)$ into another matrix pencil $D'(s)$:

$$(1)\ D'(s) = U(s)D(s)V(s), \quad (2)\ D'(s) = UD(s)V(s), \quad (3)\ D'(s) = U(s)D(s)V,$$

where $U(s), V(s)$ are unimodular matrices and U, V are nonsingular constant matrices. The results of (1) in Table 1 follow from the above discussion.

TABLE 1

The invariance of structural indices under equivalence transformations, where $\surd$ represents that the indices are invariant, and — represents that the indices can be different. Here, $U(s), V(s)$ are unimodular matrices and U, V are nonsingular constant matrices.

	H_ν	K_ρ	N_μ	L_ϵ	$L_\eta^\top$
(1) $D(s) \to U(s)D(s)V(s)$	$\surd$	$\surd$	—	—	—
(2) $D(s) \to UD(s)V(s)$	$\surd$	$\surd$	—	—	$\surd$
(3) $D(s) \to U(s)D(s)V$	$\surd$	$\surd$	—	$\surd$	—

We now consider the L_ϵ block in Table 1. Let $A(s) = \sum_{i=0}^{N} A_i s^i$ be a polynomial matrix, $U(s) = \sum_i U_i s^i$ be a unimodular matrix, and V be a nonsingular constant matrix. We denote the maximum degree of entries in $U(s)A(s)V$ by $N'(\geq N)$. Then we have $\Psi_k^{N'}(U(s)A(s)V) = \tilde{U}_{k+N'-1}\Psi_k^{N'}(A(s))\tilde{V}_k$, where $\tilde{U}_{k+N'-1}$ is a $(k+N')m \times (k+N')m$ matrix and $\tilde{V}_k$ is a $kn \times kn$ matrix defined by

$$\tilde{U}_{k+N'-1} = \begin{pmatrix} U_0 & O & \cdots & O \\ U_1 & U_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ U_{k+N'-1} & \cdots & U_1 & U_0 \end{pmatrix}, \quad \tilde{V}_k = \begin{pmatrix} V & O & \cdots & O \\ O & V & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & V \end{pmatrix}.$$

We note that $U(s)A(s)V$ does not have entries with degree $N'+1, N'+2, \ldots, N'+k-1$, because N' is the maximum degree of entries in $U(s)A(s)V$. Since $U(s)$ is unimodular, U_0 is nonsingular, which implies that $\tilde{U}_{k+N'-1}$ is nonsingular. In addition, $\tilde{V}$ is also nonsingular by the nonsingularity of V. Hence we obtain

$$\psi_k(U(s)A(s)V) = \psi_k(A(s)). \tag{3.1}$$

Let $D'(s) = U(s)D(s)V$ be a matrix pencil described in (3) in Table 1. By (3.1), we have $\psi_k(D') = \psi_k(D)$. Thus, $D(s)$ and $D'(s)$ have the same minimal column indices by (2.8). For (2) in Table 1, we can prove that $D(s)$ and $UD(s)V(s)$ have the same minimal row indices in a similar way. Thus, we complete Table 1.

4. Mixed matrix pencils and LM-matrix pencils. Let $\mathbf{K}$ be a subfield of a field $\mathbf{F}$. A typical setting of $(\mathbf{K}, \mathbf{F})$ is that $\mathbf{K}$ and $\mathbf{F}$ are the fields of rational and real numbers. A subset $\mathcal{Y} = \{y_1, \ldots, y_h\}$ of $\mathbf{F}$ is said to be *algebraically independent* over $\mathbf{K}$ if there exists no nontrivial polynomial $p(X_1, \ldots, X_h)$ over $\mathbf{K}$ such that $p(y_1, \ldots, y_h) = 0$, where $p(X_1, \ldots, X_h)$ is called nontrivial if some of its coefficients are not zero.

A matrix $A(s)$ is called a *mixed polynomial matrix* with respect to $(\mathbf{K}, \mathbf{F})$ if $A(s)$ is given by $A(s) = Q(s) + T(s)$ with a pair of polynomial matrices $Q(s)$ over $\mathbf{K}$ and $T(s)$ over $\mathbf{F}$ that satisfy the following two conditions.

(MP-Q) The coefficients of nonzero entries of $Q(s)$ belong to $\mathbf{K}$.

(MP-T) The coefficients of nonzero entries of $T(s)$ belong to $\mathbf{F}$, and the set of nonzero coefficients of $T(s)$ is algebraically independent over $\mathbf{K}$.

If $A(s)$ is expressed as $\binom{Q(s)}{T(s)}$, $A(s)$ is called a *layered mixed polynomial matrix* (*LM-polynomial matrix*).

In order to reflect the dimensional consistency in conservation laws of dynamical systems, Murota [22] introduces the following condition on $Q(s)$, which is a formal version of (DC) in section 1.

(MP-DC) Every nonvanishing subdeterminant of $Q(s)$ is a monomial in s over $\mathbf{K}$.

Let $\mathrm{diag}[a_1, a_2, \ldots, a_h]$ denote a diagonal matrix having diagonal entries $a_1, a_2, \ldots, a_h$. It is known [22, 24] that an $m \times n$ matrix $Q(s)$ satisfies (MP-DC) if and only if

$$Q(s) = \mathrm{diag}[s^{p_1}, s^{p_2}, \ldots, s^{p_m}] \cdot Q(1) \cdot \mathrm{diag}[s^{-q_1}, s^{-q_2}, \ldots, s^{-q_n}] \tag{4.1}$$

for some integers p_i $(i = 1, \ldots, m)$ and q_j $(j = 1, \ldots, n)$.

We call a mixed polynomial matrix $A(s)$ satisfying (MP-DC) a *dimensionally consistent mixed polynomial matrix* (*DCM-polynomial matrix*). If $A(s)$ is an LM-polynomial matrix, we call it a *dimensionally consistent LM-polynomial matrix* (*DCLM-polynomial matrix*).

A mixed polynomial matrix and an LM-polynomial matrix are called a *mixed matrix pencil* and an *LM-matrix pencil* if the degree of each entry is at most one. If they satisfy (MP-DC), we call them a *dimensionally consistent mixed matrix pencil* (*DCM-matrix pencil*) and a *dimensionally consistent LM-matrix pencil* (*DCLM-matrix pencil*), respectively.

Several efficient algorithms have been developed [17, 27, 36] for computing $\delta_k(A(s))$ of a mixed matrix pencil $A(s)$. If $A(s)$ is an $m \times n$ DCM-matrix pencil, the computation of $\delta_k(A(s))$ is reduced to a weighted matroid intersection problem [28, Remark 6.2.10], which can be solved with $\mathrm{O}(m^2nk)$ arithmetic operations over $\mathbf{K}$. With the aid of the fast matrix multiplication, one can improve this bound to $\mathrm{O}(m^{\omega-1}nk)$, where $\omega < 2.38$.

Example 4.1. Let $D(s) = \binom{Q(s)}{T(s)}$ be a DCLM-matrix pencil with respect to $(\mathbb{Q}, \mathbb{R})$ defined by

$$\begin{pmatrix} Q(s) \\ \hline T(s) \end{pmatrix} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & s & 1 \\ \hline -t_3 & 0 & t_1 & 0 & 0 \\ 0 & -t_4 & 0 & 0 & t_2 s \end{array}\right), \tag{4.2}$$

where $\mathcal{Y} = \{t_1, t_2, t_3, t_4\} \subseteq \mathbb{R}$ is algebraically independent over $\mathbb{Q}$. We can see that (MP-DC) is fulfilled as $Q(s) = \mathrm{diag}[1, s] \cdot Q(1) \cdot \mathrm{diag}[1, s^{-1}, 1, 1, s^{-1}]$.

Let $D_{\mathrm{M}}(s) = s(X_Q + X_T) + (Y_Q + Y_T)$ be an $m \times n$ mixed matrix pencil with $Q(s) = sX_Q + Y_Q$ and $T(s) = sX_T + Y_T$. Consider an LM-matrix pencil

$$D(s) = \begin{pmatrix} I & sX_Q + Y_Q \\ -Z & sX_T + Y_T \end{pmatrix}, \tag{4.3}$$

where Z is a diagonal matrix with the (i, i) entry being a new parameter $t_i \in \mathbf{F}$. The following corollary shows the relation between $D_{\mathrm{M}}(s)$ and $D(s)$.

COROLLARY 4.2. *The minimal row indices of* $D_{\mathrm{M}}(s)$ *coincide with those of* $D(s)$.

Proof. Let $T_{ij}(s)$ denote the (i, j) entry of $T(s) = sX_T + Y_T$. We define $\hat{D}(s) = \begin{pmatrix} I & sX_Q + Y_Q \\ -I & sX_T + Y_Y \end{pmatrix}$, which is obtained by dividing $(m + i)$th row of $D(s)$ by t_i and redefining $T_{ij}(s)/t_i$ to be $T_{ij}(s)$. Since the Kronecker canonical form is invariant under this transformation, $\hat{D}(s)$ and $D(s)$ have the same Kronecker canonical form. Let us define a nonsingular constant matrix U and a unimodular matrix $V(s)$ by $U = \begin{pmatrix} I & O \\ I & I \end{pmatrix}$ and $V(s) = \begin{pmatrix} I & -(sX_Q + Y_Q) \\ O & I \end{pmatrix}$. Then we have $U\hat{D}(s)V(s) = \begin{pmatrix} I & O \\ O & D_{\mathrm{M}}(s) \end{pmatrix}$. This transformation corresponds to (2) in Table 1. Hence, $D(s)$ and $U\hat{D}(s)V(s)$ have the same minimal row indices. The Kronecker canonical form of $U\hat{D}(s)V(s)$ consists of m copies of N_1 and the Kronecker canonical form of $D_{\mathrm{M}}(s)$. Therefore, $D(s)$ and $D_{\mathrm{M}}(s)$ have the same minimal row indices. ∎

According to (2) in Table 1, N_μ and L_ϵ blocks of $D_{\mathrm{M}}(s)$ and $D(s)$ can be different. However, their sum has the following relation.

THEOREM 4.3. *Let us denote the structural indices of* $D_{\mathrm{M}}(s)$ *and* $D(s)$ *by* $(\nu', \rho'_1, \ldots, \rho'_{c'}, \mu'_1, \ldots, \mu'_{d'}, \epsilon'_1, \ldots, \epsilon'_{p'}, \eta'_1, \ldots, \eta'_{q'})$ *and* $(\nu, \rho_1, \ldots, \rho_c, \mu_1, \ldots, \mu_d, \epsilon_1, \ldots, \epsilon_p, \eta_1, \ldots, \eta_q)$, *respectively. Then we have*

$$m + \sum_{i=1}^{d'} {\mu'}_i + \sum_{i=1}^{p'} {\epsilon'}_i = \sum_{i=1}^{d} \mu_i + \sum_{i=1}^{p} \epsilon_i.$$

Proof. As shown in the proof of Corollary 4.2, the transformation from a mixed matrix pencil into the associated LM-matrix pencil is regarded as the transformation (2) in Table 1. Hence we have $c' = c$, $q' = q$, $\nu' = \nu$, $\rho'_i = \rho_i$ $(i = 1, \ldots, c)$, and $\eta'_i = \eta_i$ $(i = 1, \ldots, q)$. Let r' and r denote the ranks of $D_{\mathrm{M}}(s)$ and $D(s)$, respectively. Due to the proof of Corollary 4.2, $r = \operatorname{rank} \hat{D}(s) = \operatorname{rank}(\begin{smallmatrix} I & O \\ O & D_{\mathrm{M}}(s) \end{smallmatrix}) = m + r'$ holds. It follows from (2.1) that $\sum_{i=1}^{d} \mu_i + \sum_{i=1}^{p} \epsilon_i = m + \sum_{i=1}^{d'} {\mu'}_i + \sum_{i=1}^{p'} {\epsilon'}_i$. ∎

5. The combinatorial canonical form in mixed matrix theory. In this section, we expound the CCF in mixed matrix theory [26, 30]. In particular, we focus on a DCLM-polynomial matrix $D(s) = \binom{Q(s)}{T(s)}$.

An *LM-admissible transformation* is defined to be an equivalence transformation in the form of

$$P_{\mathrm{r}} \begin{pmatrix} W(s) & O \\ O & I \end{pmatrix} \begin{pmatrix} Q(s) \\ T(s) \end{pmatrix} P_{\mathrm{c}}, \tag{5.1}$$

where P_{r} and P_{c} are permutation matrices, and $W(s)$ is a unimodular matrix. We remark that the resulting matrix is an LM-polynomial matrix but is not necessarily a matrix pencil even if $D(s) = \binom{Q(s)}{T(s)}$ is an LM-matrix pencil.

We denote the row index set and the column index set of $D(s) = \binom{Q(s)}{T(s)}$ by R and C, and the row index sets of $Q(s)$ and $T(s)$ by R_Q and R_T. Consider a set function $\sigma : 2^C \to \mathbb{Z}$ defined by

$$\sigma(J) = \operatorname{rank} Q(s)[R_Q, J] + \left| \bigcup_{j \in J} \{ i \in R_T \mid T_{ij}(s) \neq 0 \} \right| - |J|,$$

where $T_{ij}(s)$ denotes the (i, j) entry of $T(s)$. Then the set function σ is known to be submodular, that is, $\sigma(J_1) + \sigma(J_2) \geq \sigma(J_1 \cup J_2) + \sigma(J_1 \cap J_2)$ holds for $J_1, J_2 \subseteq C$. The family of minimizers $\mathcal{L}_{\min}(\sigma) = \{ J \subseteq C \mid \sigma(J) \leq \sigma(J') \ \forall J' \subseteq C \}$ forms a sublattice of 2^C, i.e., $J_1, J_2 \in \mathcal{L}_{\min}(\sigma)$ implies $J_1 \cup J_2, J_1 \cap J_2 \in \mathcal{L}_{\min}(\sigma)$.

Let $\mathcal{C} : J_0 \subsetneq J_1 \subsetneq \cdots \subsetneq J_b$ be a maximal chain in $\mathcal{L}_{\min}(\sigma)$. Put $C_0 = J_0$, $C_k = J_k \setminus J_{k-1}$ for $k = 1, \ldots, b$, and $C_\infty = C \setminus J_b$ to obtain a partition $(C_0; C_1, \ldots, C_b; C_\infty)$ of C. Based on this partition, $D(s)$ can be brought into the CCF by an LM-admissible transformation as follows.

Theorem 5.1 (see [26, Lemma 3.1]). *Let $D(s) = \binom{Q(s)}{T(s)}$ be a DCLM-polynomial matrix. By an LM-admissible transformation, $D(s)$ can be brought into another LM-polynomial matrix $\tilde{D}(s) = \binom{\tilde{Q}(s)}{\tilde{T}(s)}$ with the following properties.*

- (B1) *$\tilde{D}(s)$ is block-triangularized, i.e., $\tilde{D}[R_k, C_l] = O$ if $0 \leq l < k \leq \infty$ with respect to partitions $(R_0; R_1, \ldots, R_b; R_\infty)$ and $(C_0; C_1, \ldots, C_b; C_\infty)$ of the row/column index sets of $\tilde{D}(s)$, where $b \geq 0$, $R_k \neq \emptyset$, and $C_k \neq \emptyset$ for $k = 1, \ldots, b$, and R_0, R_∞, C_0, and C_∞ can be empty.*
- (B2) *The sizes of the diagonal blocks satisfy* (i) $|R_0| < |C_0|$ *or* $|R_0| = |C_0| = 0$, (ii) $|R_k| = |C_k| > 0$ *for* $k = 1, \ldots, b$, (iii) $|R_\infty| > |C_\infty|$ *or* $|R_\infty| = |C_\infty| = 0$.
- (B3) *The diagonal blocks satisfy* (i) $\operatorname{rank} \tilde{D}[R_0, C_0] = |R_0|$, (ii) $\operatorname{rank} \tilde{D}[R_k, C_k] = |R_k| = |C_k|$ *for* $k = 1, \ldots, b$, (iii) $\operatorname{rank} \tilde{D}[R_\infty, C_\infty] = |C_\infty|$.

(B4) *The diagonal blocks also satisfy* (i) $\operatorname{rank} \tilde{D}[R_0, C_0 \setminus \{j\}] = |R_0|$ $(j \in C_0)$, (ii) $\operatorname{rank} \tilde{D}[R_k \setminus \{i\}, C_k \setminus \{j\}] = |R_k| - 1 = |C_k| - 1$ $(i \in R_k, j \in C_k)$ *for* $k = 1, \ldots, b$, (iii) $\operatorname{rank} \tilde{D}[R_\infty \setminus \{i\}, C_\infty] = |C_\infty|$ $(i \in R_\infty)$.

(B5) $\tilde{D}(s)$ *is the finest proper block-triangular matrix among LM-polynomial matrices connected by an LM-admissible transformation.*

We call $\tilde{D}(s)$ in Theorem 5.1 the *CCF* of $D(s)$, and $D_0(s) := \tilde{D}[R_0, C_0]$ the *horizontal tail*.

Recall the definition of $d_k(A(s))$ in (2.7). We now have the following lemma.

LEMMA 5.2 (see [28, Theorem 6.3.4 and Remark 6.3.7]). *Let* $D(s) = \binom{Q(s)}{T(s)}$ *be a DCLM-polynomial matrix of rank* r. *The* r*th monic determinantal divisor* $d_r(D(s))$ *can be expressed by* $d_r(D(s)) = \alpha_r \cdot g(s) \cdot \prod_{l=1}^{b} \det \tilde{D}(s)[R_l, C_l]$, *where* $\alpha_r \in \mathbf{F}$ *is a constant,* $g(s)$ *is a monomial in* s, *and* $\tilde{D}(s)[R_l, C_l]$ $(l = 1, \ldots, b)$ *are the square blocks which appear in the CCF of* $D(s)$.

If $D(s)$ is a DCLM-matrix pencil, we can construct a CCF such that the horizontal tail $D_0(s)$ is also a DCLM-matrix pencil. In the expression (4.1) of $Q(s)$, we can assume $p_1 \le p_2 \le \cdots \le p_m$ and $q_1 \le q_2 \le \cdots \le q_n$ without loss of generality. We now briefly describe the algorithm for computing $D_0(s)$, which is given in [26, §3.2].

Step 1. Determine the partition $(C_0; C_1, \ldots, C_b; C_\infty)$ of C with reference to the set function σ by solving a matroid intersection problem.
Step 2. Find a basis of the row vectors of the submatrix $Q(1)[R_Q, C_0]$ by collecting independent row vectors according to the ordering with reference to p_i in such a manner that $p_1 \le p_2 \le \cdots \le p_m$. This ordering guarantees that $W(s)$ of (5.1) is a unimodular matrix. We denote the basis by R_{Q0}.
Step 3. Output $R_0 = R_{Q0} \cup R_{T0}$ and C_0, where $R_{T0} = \{i \in R_T \mid T_{ij}(s) \ne 0, j \in C_0\}$.

The bottleneck part is Step 1, which requires $\mathrm{O}(n^3 \log n)$ arithmetic operations [5] over $\mathbf{K}$, where $n = |C|$ and we assume $|R| = \mathrm{O}(n)$ for simplicity. The complexity can be improved to $\mathrm{O}(n^{2.62})$ by adopting the algorithm of Gabow and Xu [9].

In Step 2, we have assumed that an ordering of rows h and h' with $p_h = p_{h'}$ is arbitrary. By determining this ordering based on $q_1, q_2, \ldots, q_n$, we prove the following lemma.

LEMMA 5.3. *If* $D(s) = \binom{Q(s)}{T(s)}$ *is a DCLM-matrix pencil, one can construct a CCF of* $D(s)$ *such that the horizontal tail* $D_0(s)$ *is also a DCLM-matrix pencil.*

Proof. Since $Q(s)$ is a matrix pencil, $Q(1)[R_Q, C_0]$ is in the form of

$$Q^0 = \begin{array}{c} \\ \mathrm{Row}(0) \\ \mathrm{Row}(1) \\ \mathrm{Row}(2) \\ \vdots \\ \vdots \\ \mathrm{Row}(\gamma) \end{array} \begin{array}{c} \begin{array}{ccccc} \mathrm{Col}(0) & \mathrm{Col}(1) & \cdots & \mathrm{Col}(\gamma-2) & \mathrm{Col}(\gamma-1) \end{array} \\ \left(\begin{array}{ccccc} * & O & \cdots & \cdots & O \\ ** & * & \ddots & & \vdots \\ O & ** & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * & O \\ \vdots & & \ddots & ** & * \\ O & \cdots & \cdots & O & ** \end{array}\right) \end{array}$$

for some γ, where $\mathrm{Row}(h) = \{i \in R_Q \mid p_i = h\}$ and $\mathrm{Col}(h) = \{j \in C_0 \mid q_j = h\}$. Here, $*$ and $**$ denote a constant matrix and a coefficient matrix of s, respectively.

We can find a basis of the row vectors of the submatrix $Q^0[R_Q, C_0]$ by collecting independent row vectors from the top row to the bottom row, as explained below. We first find $R_*^0 \subseteq \mathrm{Row}(0)$ satisfying $\mathrm{rank}\, Q^0[R_*^0, \mathrm{Col}(0)] = \mathrm{rank}\, Q^0[\mathrm{Row}(0), \mathrm{Col}(0)]$, which means that R_*^0 is a basis of $\mathrm{Row}(0)$. By row transformations, we obtain Q^1 from Q^0 such that

$$Q^1[\mathrm{Row}(0)\cup\mathrm{Row}(1), C_0] = \begin{array}{l} \\ R_*^0 \\ \mathrm{Row}(0)\setminus R_*^0 \\ \mathrm{Row}(1)\end{array} \begin{array}{c} \leftarrow \quad \mathrm{Col}(0) \quad \rightarrow \quad \mathrm{Col}(1) \quad \cdots \\ \left(\begin{array}{ccccc} I & * & * & O & O \\ O & O & O & O & O \\ O & ** & ** & * & O \end{array}\right) \end{array},$$

because the row vectors of $Q^0[\mathrm{Row}(0)\setminus R_*^0, C_0]$ can be expressed as linear combinations of those of $Q^0[R_*^0, C_0]$.

Next, we find $R_{**}^1 \subseteq \mathrm{Row}(1)$ satisfying $\mathrm{rank}\, Q^1[R_{**}^1, \mathrm{Col}(0)] = \mathrm{rank}\, Q^1[\mathrm{Row}(1), \mathrm{Col}(0)]$. Then we obtain Q^2 from Q^1 such that

$$Q^2[\mathrm{Row}(0)\cup\mathrm{Row}(1), C_0] = \begin{array}{l} \\ R_*^0 \\ \mathrm{Row}(0)\setminus R_*^0 \\ R_{**}^1 \\ \mathrm{Row}(1)\setminus R_{**}^1 \end{array} \begin{array}{c} \leftarrow \quad \mathrm{Col}(0) \quad \rightarrow \quad \mathrm{Col}(1) \quad \cdots \\ \left(\begin{array}{ccccc} I & * & * & O & O \\ O & O & O & O & O \\ O & I & ** & * & O \\ O & O & O & * & O \end{array}\right) \end{array}$$

by row transformations. Then, we apply the same procedure to $Q^2[R_Q \setminus (\mathrm{Row}(0) \cup R_{**}^1), C_0 \setminus \mathrm{Col}(0)]$.

As a result, we obtain

$$Q' = \begin{array}{c} \\ \mathrm{Row}(0) \\ \mathrm{Row}(1) \\ \mathrm{Row}(2) \\ \vdots \\ \vdots \\ \mathrm{Row}(\gamma) \end{array} \begin{array}{c} \begin{array}{ccccc} \mathrm{Col}(0) & \mathrm{Col}(1) & \cdots & \mathrm{Col}(\gamma-2) & \mathrm{Col}(\gamma-1) \end{array} \\ \left(\begin{array}{ccccc}
\begin{smallmatrix} I & * & * \\ O & O & O \end{smallmatrix} & O & \cdots & \cdots & O \\
\begin{smallmatrix} O & I & ** \\ O & O & O \\ O & O & O \end{smallmatrix} & \begin{smallmatrix} * & * & * \\ I & * & * \\ O & O & O \end{smallmatrix} & \ddots & & \vdots \\
O & \begin{smallmatrix} O & I & ** \\ O & O & O \end{smallmatrix} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \begin{smallmatrix} * & * & * \\ I & * & * \\ O & O & O \end{smallmatrix} & O \\
\vdots & & \ddots & \begin{smallmatrix} O & I & ** \\ O & O & O \\ O & O & O \end{smallmatrix} & \begin{smallmatrix} * & * & * \\ I & * & * \\ O & O & O \end{smallmatrix} \\
O & \cdots & \cdots & O & \begin{smallmatrix} O & I & ** \\ O & O & O \end{smallmatrix}
\end{array}\right) \end{array},$$

where the row index sets of I in $Q'[\mathrm{Row}(h), \mathrm{Col}(h)]$ and $Q'[\mathrm{Row}(h), \mathrm{Col}(h-1)]$ correspond to R_*^h and R_{**}^h, respectively. This transformation preserves $Q'[\mathrm{Row}(h), \mathrm{Col}(l)] = O$ for any h, l satisfying $0 \le h \le \gamma$, $0 \le l \le \gamma - 1$, and $h - l \neq 0, 1$. We define $R_{Q0} = \bigcup_{i=1}^{\gamma}(R_*^{i-1} \cup R_{**}^i)$. Then R_{Q0} is a basis of the row vectors of Q' as well as $Q(1)[R_Q, C_0] = Q^0$.

Let W be a nonsingular constant matrix such that $Q' = WQ^0$. We define $W(s)$ in (5.1) by $W(s) = \mathrm{diag}[s^{p_1}, s^{p_2}, \ldots, s^{p_m}] \cdot W \cdot \mathrm{diag}[s^{-p_1}, s^{-p_2}, \ldots, s^{-p_m}]$. The horizontal tail $D_0(s)$ is given by

$$D_0(s) = \begin{pmatrix} W(s)Q(s)[R_{Q0}, C_0] \\ T(s)[R_{T0}, C_0] \end{pmatrix},$$

where R_{T0} is defined in Step 3 in the algorithm for computing $D_0(s)$.

To prove that $D_0(s)$ is a matrix pencil, it suffices to show that $W(s)Q(s)[R_{Q0}, C_0]$ is also a matrix pencil, because $T(s)$ is a matrix pencil. We now have

$$W(s)Q(s) = \mathrm{diag}[s^{p_1}, s^{p_2}, \dots, s^{p_m}] \cdot WQ(1) \cdot \mathrm{diag}[s^{-q_1}, s^{-q_2}, \dots, s^{-q_n}]$$

and $WQ(1)[R_Q, C_0] = WQ^0 = Q'$. Hence $W(s)Q(s)[R_Q, C_0]$ is a matrix pencil, which implies that the submatrix $W(s)Q(s)[R_{Q0}, C_0]$ of $W(s)Q(s)[R_Q, C_0]$ is also a matrix pencil. Moreover, $D_0(s)$ satisfies (MP-DC), because $W(s)Q(s)$ satisfies (MP-DC). □

Example 5.4. Consider a DCLM-matrix pencil

$$D(s) = \begin{pmatrix} 1 & 1 & 0 & s \\ s & s & 1 & 0 \\ \hline 0 & 0 & t_1 s & t_2 \end{pmatrix}$$

with respect to $(\mathbb{Q}, \mathbb{R})$, where $\{t_1, t_2\} \subseteq \mathbb{R}$ is algebraically independent over $\mathbb{Q}$. By setting $W(s) = \left(\begin{smallmatrix} 1 & 0 \\ -s & 1 \end{smallmatrix}\right)$, we obtain the CCF represented as

$$\left(\begin{array}{cc|cc} 1 & 1 & 0 & s \\ \hline 0 & 0 & 1 & -s^2 \\ 0 & 0 & t_1 s & t_2 \end{array}\right).$$

The horizontal tail remains a matrix pencil, while the square block is not.

6. The Kronecker canonical form via CCF. In this section, we investigate the Kronecker canonical form of DCLM-matrix pencils via the CCF decomposition. For a DCLM-matrix pencil $D(s) = \binom{Q(s)}{T(s)}$ of rank r, we construct its CCF $\tilde{D}(s)$ so that the horizontal tail $D_0(s)$ is also a DCLM-matrix pencil. The existence of such CCF is assured by Lemma 5.3. The rank of $D_0(s)$ is denoted by r_0.

LEMMA 6.1. *We have* $\psi_k(D) = \psi_k(D_0) + k(r - r_0)$.

Proof. We define $D_*(s) = \tilde{D}(s)[R \setminus R_0, C \setminus C_0]$. Since $D_*(s)$ is of full-column rank, it holds that $\psi_k(D_*) = k|C \setminus C_0| = k(r - r_0)$ by Lemma 2.3. We also have $\psi_k(\tilde{D}) = \psi_k(D_0) + \psi_k(D_*)$ by $\tilde{D}(s)[R \setminus R_0, C_0] = O$. By (3.1) and the definition of an LM-admissible transformation (5.1), $\psi_k(\tilde{D}) = \psi_k(D)$ holds. Thus we obtain $\psi_k(D) = \psi_k(D_0) + k(r - r_0)$. □

We now investigate the Kronecker canonical form of $D_0(s)$.

LEMMA 6.2. *The monic determinantal divisor* $d_{r_0}(D_0(s))$ *is a monomial in* s.

Proof. By $\mathrm{rank}\, D_0(s) = r_0$, we can apply Lemma 5.2 to $d_{r_0}(D_0(s))$. Since the CCF of $D_0(s)$ has no square blocks, $d_{r_0}(D_0(s))$ is a monomial in s by Lemma 5.2. □

We obtain the following theorem on the sum of the minimal column indices.

THEOREM 6.3. *The sum of the minimal column indices of a DCLM-matrix pencil* $D(s)$ *is obtained by*

$$\sum_{i=1}^{p} \epsilon_i = \delta_{r_0}(D_0(s)) - \zeta_{r_0}(D_0(s)). \tag{6.1}$$

Proof. The horizontal tail $D_0(s)$ has the Kronecker canonical form, because $D_0(s)$ is a matrix pencil by Lemma 5.3. By (2.8) and Lemma 6.1, the minimal column indices of $D(s)$ coincide with those of the horizontal tail $D_0(s)$. Let $\bar{D}_0(s)$ be the Kronecker canonical form of $D_0(s)$. Since $D_0(s)$ is of full-row rank, $\bar{D}_0(s)$ contains no rectangular blocks $L_\eta^\top$. In addition, $\bar{D}_0(s)$ does not contain a strictly regular block by Lemma 6.2. Hence we obtain (6.1) by (2.6). ◻

Theorem 6.3 indicates that the computation of $\sum_{i=1}^p \epsilon_i$ for a DCLM-matrix pencil $D(s)$ reduces to that of $\delta_{r_0}(D_0(s))$ and $\zeta_{r_0}(D_0(s))$. We now discuss the time complexity of computing $\sum_{i=1}^p \epsilon_i$ of an $m \times n$ DCLM-matrix pencil $D(s)$, assuming $m = O(n)$ for simplicity. Recall that the CCF of $D(s)$ can be found in $\mathrm{O}(n^{2.62})$ time. Then one can compute $\delta_{r_0}(D_0(s))$ and $\zeta_{r_0}(D_0(s))$ in $\mathrm{O}(r_0^{2.38} n_0)$ time, where $n_0 = |C_0|$. Thus the total running time bound is $\mathrm{O}(n^{2.62} + r_0^{2.38} n_0)$. Note that r_0 and n_0 are smaller (and sometimes much smaller) than n.

Theorem 6.3 combined with Theorem 4.3 enables us to compute the sum of the minimal column indices of a DCM-matrix pencil, as explained below. Let $D_{\mathrm{M}}(s) = s(X_Q + X_T) + (Y_Q + Y_T)$ be an $m \times n$ DCM-matrix pencil and $D(s)$ its associated LM-matrix pencil defined by (4.3). We denote the structural indices of $D_{\mathrm{M}}(s)$ by $(\nu', \rho'_1, \ldots, \rho'_{c'}, \mu'_1, \ldots, \mu'_{d'}, \epsilon'_1, \ldots, \epsilon'_{p'}, \eta'_1, \ldots, \eta'_{q'})$. It follows from Theorem 4.3 that $\sum_{i=1}^{p'} \epsilon'_i = \sum_{i=1}^{p} \epsilon_i + \sum_{i=1}^{d} \mu_i - \sum_{i=1}^{d'} \mu'_i - m$. In the right-hand side, $\sum_{i=1}^p \epsilon_i$ of the LM-matrix pencil $D(s)$ can be computed by Theorem 6.3. We can also find $\sum_{i=1}^d \mu_i$ of $D(s)$ and $\sum_{i=1}^{d'} \mu'_i$ of $D_{\mathrm{M}}(s)$ based on (2.3), because δ_k is computed efficiently as explained in section 4. It should be noted that, in the computation of δ_k, the transformation from a mixed matrix pencil into an LM-matrix pencil is different from (4.3). Thus we can obtain $\sum_{i=1}^{p'} \epsilon'_i$ of the DCM-matrix pencil $D_{\mathrm{M}}(s)$.

In order to compute the sum of the minimal row indices, we apply Theorems 4.3 and 6.3 to $D(s)^\top$, because the minimal row indices of $D(s)$ coincide with the minimal column indices of $D(s)^\top$.

We conclude this section with an example.

Example 6.4. Consider a DCM-matrix pencil $D_{\mathrm{M}}(s) = \left(\begin{smallmatrix} t_1 & 0 & 0 \\ 0 & s & t_2 s+1 \end{smallmatrix}\right)$ with respect to $(\mathbb{Q}, \mathbb{R})$, where $\{t_1, t_2\} \subseteq \mathbb{R}$ is algebraically independent over $\mathbb{Q}$. Its associated LM-matrix pencil $D(s)$ is given by (4.2). The Kronecker canonical forms of $D_{\mathrm{M}}(s)$ and $D(s)$ are in the forms of block-diag(N_1, L_1) and block-diag(N_1, N_1, L_2), respectively. As described below, we can find the sums of the minimal column indices without computing the Kronecker canonical form, based on Theorems 4.3 and 6.3.

We first find the horizontal tail $D_0(s) = \left(\begin{smallmatrix} 1 & s & 1 \\ -t_4 & 0 & t_2 s \end{smallmatrix}\right)$ in the CCF of $D(s)$ by the procedure in section 5. Then it follows from Theorem 6.3 that

$$\sum_i \epsilon_i = \delta_2(D_0(s)) - \zeta_2(D_0(s)) = 2 - 0 = 2.$$

We can obtain $\sum_i \mu_i = 2$ and $\sum_i \mu'_i = 1$ by executing any of the algorithms given in [17, 27, 36] or reducing to a weighted matroid intersection problem [28, Remark 6.2.10]. Thus it follows from Theorem 4.3 that $\sum_i \epsilon'_i = 2 + 2 - 1 - 2 = 1$.

7. Application to controllable subspace. Consider a linear time-invariant dynamical system in a descriptor form

$$F\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t), \tag{7.1}$$

where F and A are $n \times n$ matrices and B is an $n \times l$ matrix. For the unique solvability, we assume that $A - sF$ is a regular matrix pencil. In this section, we present an application of our main result to controllability analysis of (7.1) with $\big(A - sF \mid B\big)$ being a DCM-matrix pencil.

Van Dooren [41] introduced the controllable subspace of (7.1) defined by

$$\mathcal{C} = \inf\{\mathcal{S} \mid \dim(F\mathcal{S} + A\mathcal{S}) = \dim \mathcal{S}, \operatorname{im} B \subseteq F\mathcal{S} + A\mathcal{S}\},$$

where the infimum can be proven to exist. In fact, the controllable subspace $\mathcal{C}$ is obtained as follows. With an appropriate nonsingular constant matrix S, one can transform $\big(A - sF \mid B\big)$ into

$$S\left(A - sF \mid B\right) = \begin{pmatrix} A_0 - sF_0 & O \\ * & B_0 \end{pmatrix}$$

so that B_0 is of full-row rank. Since $A_0 - sF_0$ is of full-row rank, its Kronecker canonical form does not contain $L_\eta^\top$ blocks. Therefore one can further transform $A_0 - sF_0$ with an appropriate pair of nonsingular constant matrices U and V into

$$U\left(A_0 - sF_0\right)V = \begin{pmatrix} A_1 - sF_1 & O \\ O & A_2 - sF_2 \end{pmatrix},$$

where $A_1 - sF_1$ is regular and the Kronecker canonical form of $A_2 - sF_2$ consists only of L_ϵ blocks. Then the column index set of $A_2 - sF_2$ corresponds to the controllable subspace $\mathcal{C}$, and the number of columns is equal to $\dim \mathcal{C}$.

The system (7.1) is controllable iff $\dim \mathcal{C} = n$. Murota [23] presented a matroid-theoretic algorithm for testing the controllability of a dynamical system (7.1) described by a DCM-matrix pencil $\big(A - sF \mid B\big)$. The algorithm, however, does not provide the dimension of the controllable subspace.

The following lemma shows that $\dim \mathcal{C}$ is characterized by the rank of the $(n+1)n \times (n^2 + nl + l)$ matrix

$$\Sigma(F, A, B) = \begin{pmatrix} B & -F & O & O & \cdots & O & O & O & O \\ O & A & B & -F & \ddots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & A & \ddots & O & O & O & O \\ O & O & O & O & \ddots & -F & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & A & B & -F & O \\ O & O & O & O & \cdots & O & O & A & B \end{pmatrix}.$$

LEMMA 7.1. *It holds that* $\dim \mathcal{C} = \operatorname{rank} \Sigma(F, A, B) - n^2$.

Proof. We denote the row index sets of $A_1 - sF_1$, $A_2 - sF_2$, and B_0 by R_1, R_2, and R_3. Since $A_1 - sF_1$ is a regular matrix pencil, we have $\dim \mathcal{C} = n - |R_1|$.

The rank of $\Sigma(F, A, B)$ is invariant under the above equivalence transformation. By putting $\hat{A} = \left(\begin{smallmatrix} A_1 & O \\ O & A_2 \\ * & * \end{smallmatrix}\right)$, $\hat{F} = \left(\begin{smallmatrix} F_1 & O \\ O & F_2 \\ * & * \end{smallmatrix}\right)$, and $\hat{B} = \left(\begin{smallmatrix} O \\ O \\ B_0 \end{smallmatrix}\right)$, we have $\operatorname{rank} \Sigma(F, A, B) = \operatorname{rank} \Sigma(\hat{F}, \hat{A}, \hat{B})$. Since B_0 is of full-row rank, we have

$$\begin{aligned} \operatorname{rank} \Sigma(\hat{F}, \hat{A}, \hat{B}) &= (n+1)|R_3| + \operatorname{rank} \Psi_n\left(\begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix} - s\begin{pmatrix} F_1 & O \\ O & F_2 \end{pmatrix}\right) \\ &= (n+1)|R_3| + \operatorname{rank} \begin{pmatrix} \Psi_n(A_1 - sF_1) & O \\ O & \Psi_n(A_2 - sF_2) \end{pmatrix}. \end{aligned}$$

Since $A_1 - sF_1$ is regular, it follows from Lemma 2.3 that $\psi_n(A_1 - sF_1) = n|R_1|$. By (2.8), we have $\psi_n(A_2 - sF_2) = n|R_2| + \sum_{i=1}^{p'} \min\{n, \epsilon'_i\} = n|R_2| + \sum_{i=1}^{p'} \epsilon'_i$, where $\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_{p'}$ denote the minimal column indices of $A_2 - sF_2$. Since the Kronecker canonical form of $A_2 - sF_2$ consists only of rectangular blocks L_ϵ, it holds that $\sum_{i=1}^{p'} \epsilon'_i = |R_2|$. Thus we obtain $\operatorname{rank} \Sigma(F, A, B) = (n+1)|R_3| + n|R_1| + (n+1)|R_2| = n^2 + \dim \mathcal{C}$ by $n = |R_1| + |R_2| + |R_3|$ and $\dim \mathcal{C} = n - |R_1|$. □

The following theorem states that if F is nonsingular, the computation of $\dim \mathcal{C}$ is reduced to the computation of the sum of the minimal column indices of $\big(A - sF \mid B\big)$.

THEOREM 7.2. *Let $\mathcal{C}$ be the controllable subspace of the system* (7.1)*, and* $\epsilon_1, \epsilon_2, \ldots, \epsilon_p$ *be the minimal column indices of a matrix pencil* $D(s) = \big(A - sF \mid B\big)$*. If* F *is nonsingular, the dimension of* $\mathcal{C}$ *is given by* $\dim \mathcal{C} = \sum_{i=1}^{p} \epsilon_i$.

Proof. Since we have

$$\Psi_{n+1}(D) = \begin{pmatrix} -F & O & \cdots & O \\ A & & & \\ O & & & \\ \vdots & & \Sigma(F, A, B) & \\ O & & & \end{pmatrix},$$

$\psi_{n+1}(D) = n + \operatorname{rank} \Sigma(F, A, B)$ holds by the nonsingularity of F. Then $\psi_{n+1}(D) = n(n+1) + \sum_{i=1}^{p} \min\{n+1, \epsilon_i\} = n^2 + n + \sum_{i=1}^{p} \epsilon_i$ follows from (2.8) and $\operatorname{rank} D(s) = n$. Thus we obtain $\dim \mathcal{C} = \operatorname{rank} \Sigma(F, A, B) - n^2 = \psi_{n+1}(D) - n - n^2 = \sum_{i=1}^{p} \epsilon_i$ by Lemma 7.1. □

By Theorem 7.2, if F is nonsingular, the computation of the dimension of the controllable subspace $\mathcal{C}$ is reduced to that of the sum of the minimal column indices of $D(s) = \big(A - sF \mid B\big)$. If in addition $D(s)$ is a DCM-matrix pencil, one can obtain $\dim \mathcal{C}$ by solving a weighted matroid intersection problem as described in section 6.

8. Conclusion. For mixed matrix pencils satisfying the assumption on dimensional consistency, we have characterized the sums of the minimal row/column indices of the Kronecker canonical form. An efficient matroid-theoretic algorithm for computing them is derived from this characterization. The algorithm can be used to compute the dimension of the controllable subspace in a linear time-invariant system (7.1) whose coefficient matrix F is nonsingular. An extension to the case of singular F is left for future investigation.

Our ultimate target is to present an algorithm based on structural approach for computing the minimal row/column indices. We anticipate that the characterization of the sums is useful for design of such algorithms.

REFERENCES

[1] T. BEELEN AND P. VAN DOOREN, *An improved algorithm for the computation of Kronecker's canonical form of a singular pencil*, Linear Algebra Appl., 105 (1988), pp. 9–65.

[2] T. BERGER AND T. REIS, *Controllability of linear differential-algebraic systems—A survey*, in Surveys in Differential-Algebraic Equations I, A. Ilchmann and T. Reis, eds., Springer-Verlag, Berlin, pp. 1–61, 2013.

[3] T. BERGER AND S. TRENN, *The quasi-Kronecker form for matrix pencils*, SIAM J. Matrix Anal. Appl., 33 (2012), pp. 336–368.

[4] T. BERGER AND S. TRENN, *Addition to "the quasi-Kronecker form for matrix pencils,"* SIAM J. Matrix Anal. Appl., 34 (2013), pp. 94–101.

[5] W. H. Cunningham, *Improved bounds for matroid partition and intersection algorithms*, SIAM J. Comput., 15 (1986), pp. 948–957.

[6] J. Demmel and B. Kågström, *Accurate solutions of ill-posed problems in control theory*, SIAM J. Matrix Anal. Appl., 9 (1988), pp. 126–145.

[7] J. Demmel and B. Kågström, *The generalized Schur decomposition of an arbitrary pencil $A-\lambda B$: Robust software with error bounds and applications, Part* I: *Theory and algorithms*, ACM Trans. Math. Software, 19 (1993), pp. 160–174.

[8] J. Demmel and B. Kågström, *The generalized Schur decomposition of an arbitrary pencil $A-\lambda B$: Robust software with error bounds and applications, Part* II: *Software and applications*, ACM Trans. Math. Software, 19 (1993), pp. 175–201.

[9] H. N. Gabow and Y. Xu, *Efficient theoretic and practical algorithms for linear matroid intersection problems*, J. Comput. System Sci., 53 (1996), pp. 129–147.

[10] F. R. Gantmacher, The Theory of Matrices, Chelsea, New York, 1959.

[11] K. Glover and L. M. Silverman, *Characterization of structural controllability*, IEEE Trans. Autom. Control, AC-21 (1976), pp. 534–537.

[12] E. Hairer and G. Wanner, Solving Ordinary Differential Equations II, 2nd ed., Springer-Verlag, Berlin, 1996.

[13] S. Hosoe, *Determination of generic dimensions of controllable subspaces and its application*, IEEE Trans. Automat. Control, AC-25 (1980), pp. 1192–1196.

[14] A. Ilchmann, *Time-varying linear systems and invariants of system equivalence*, Internat. J. Control, 42 (1985), pp. 759–790.

[15] S. Iwata and R. Shimizu, *Combinatorial analysis of singular matrix pencils*, SIAM J. Matrix Anal. Appl., 29 (2007), pp. 245–259.

[16] S. Iwata and M. Takamatsu, *On the Kronecker canonical form of mixed matrix pencils*, SIAM J. Matrix Anal. Appl., 32 (2011), pp. 44–71.

[17] S. Iwata and M. Takamatsu, *Computing the maximum degree of minors in mixed polynomial matrices via combinatorial relaxation*, Algorithmica, 66 (2013), pp. 346–368.

[18] B. Kågström, *RGSVD—an algorithm for computing the Kronecker structure and reducing subspaces of singular $A-\lambda B$ pencils*, SIAM J. Sci. and Stat. Comput., 7 (1986), pp. 185–211.

[19] M. Kuijper, First-order Representations of Linear Systems, Birkhäuser, Boston, 1994.

[20] P. Kunkel and V. Mehrmann, Differential-Algebraic Equations: Analysis and Numerical Solutions, European Mathematical Society, Zürich, 2006.

[21] C.-T. Lin, *Structural controllability*, IEEE Trans. Automatic Control, 19 (1974), pp. 201–208.

[22] K. Murota, *Use of the concept of physical dimensions in the structural approach to systems analysis*, Jpn. J. Appl. Math., 2 (1985), pp. 471–494.

[23] K. Murota, *Refined study on structural controllability of descriptor systems by means of matroids*, SIAM J. Control Optim., 25 (1987), pp. 967–989.

[24] K. Murota, Systems Analysis by Graphs and Matroids — Structural Solvability and Controllability, Springer-Verlag, Berlin, 1987.

[25] K. Murota, *A matroid-theoretic approach to structurally fixed modes of control systems*, SIAM J. Control Optim., 27 (1989), pp. 1381–1402.

[26] K. Murota, *On the Smith normal form of structured polynomial matrices*, SIAM J. Matrix Anal. Appl., 12 (1991), pp. 747–765.

[27] K. Murota, *On the degree of mixed polynomial matrices*, SIAM J. Matrix Anal. Appl., 20 (1999), pp. 196–227.

[28] K. Murota, Matrices and Matroids for Systems Analysis, Springer-Verlag, Berlin, 2000.

[29] K. Murota and M. Iri, *Structural solvability of systems of equations—A mathematical formulation for distinguishing accurate and inaccurate numbers in structural analysis of systems*, Jpn. J. Appl. Math., 2 (1985), pp. 247–271.

[30] K. Murota, M. Iri, and M. Nakamura, *Combinatorial canonical form of layered mixed matrices and its application to block-triangularization of systems of equations*, SIAM J. on Algebraic and Discrete Methods, 8 (1987), pp. 123–149.

[31] K. Murota and J. W. van der Woude, *Structure at infinity of structured descriptor systems and its applications*, SIAM J. Control Optim., 29 (1991), pp. 878–894.

[32] C. C. Pantelides, *The consistent initialization of differential-algebraic systems*, SIAM J. Sci. and Stat. Comput., 9 (1988), pp. 213–231.

[33] J. D. Pryce, *A Simple Structural Analysis Method for DAEs*, BIT, 41 (2001), pp. 364–394.

[34] R. Riaza, Differential-Algebraic Systems: Analytical Aspects and Circuit Applications, World Scientific, Singapore, 2008.

[35] H. H. Rosenbrock, State-space and Multivariable Theory, Nelson, London, 1970.

[36] S. Sato, *Combinatorial relaxation algorithm for the entire sequence of the maximum degree of minors in mixed polynomial matrices*, JSIAM Letters, 7 (2015), pp. 49–52.

[37] R. W. Shields and J. B. Pearson, *Structural controllability of multiinput linear systems*, IEEE Trans. Automat. Control, AC-21 (1976), pp. 203–212.
[38] J. S. Thorp, *The singular pencil of a linear dynamical system*, Internat. J. Control, 18 (1973), pp. 577–596.
[39] M. Tiller, Introduction to Physical Modeling with Modelica, Kluwer Academic Publishers, Norwell, MA, 2001.
[40] P. Van Dooren, *The computation of Kronecker's canonical form of a singular pencil*, Linear Algebra Appl., 27 (1979), pp. 103–140.
[41] P. Van Dooren, *The generalized eigenstructure problem in linear system theory*, IEEE Trans. Automat. Control, 26 (1981), pp. 111–129.
[42] W. A. Wolovich, Linear Multivariable Systems, Springer-Verlag, Berlin, 1974.
[43] W. M. Wonham, Linear Multivariable Control: A Geometric Approach, Springer-Verlag, Berlin, 1979.